Can The Spectra Of Hermitian Operator Be Invariant Under $x \leftrightarrow p$?

Case Study: 1D General Oscillator

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Abstract

We address an intriguing question on spectral invariance in quantum mechanics on exchange of co-ordinate and momentum $x \leftrightarrow p$ considering general oscillator as an example.

Keywords

exchange of co-ordinate and momentum, spectral invariance, general oscillator.

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I. Introduction

In a recent paper Rath and Mallick [1] proposed a generalised model on co-ordinate and momentum transformation in the case of Harmonic Oscillator to reflect the spectral invariance. Further it is well known that commutation relation

\[ [x, p] = i \] (1)

between co-ordinate (x) and momentum (p) on exchange($x \leftrightarrow p$) becomes

\[ [p, x] = -i \] (2)

Now question arises whether spectra of Hermitian operator (more precisely self-adjoint operator) be invariant under exchange of co-ordinate and momentum ?. If the answer to this case is yes, then why not address this to some model Hermitian operator. In this context we would like to state that in the past there was a considerable interest among many others to study spectra of anharmonic oscillator [2-9]. In any way we consider a more general type of oscillator [2-9] and study its spectra on exchange of co-ordinate and momentum.

II. New Operator and Spectra

Here we consider the operator

\[ H = \mu x^2 + \lambda_1 p^2 + \lambda_2 p^4 + \lambda_3 p^6 \] (3)

and study its spectra on exchange of co-ordinate and momentum.

\[ H \Psi = \lambda \Psi \] (4)

In order to solve it we use the eigenvalue relation [2,6,8,10,11]
where

$$|\psi> = \sum A_m|m>$$  \hspace{1cm} (6)

In the above $|m>$ stands for standard harmonic oscillator wave function\[10,11\] satisfying the relation

$$[p^2 + x^2]|m> = (2m + 1)|m>$$  \hspace{1cm} (7)

Now using the above relation one will notice that $A_m$ satisfies the following recurrence relation

$$P_m A_{m-6} + Q_m A_{m-4} + R_m A_{m} + S_m A_{m+2} + T_m A_{m+4} + U_m A_{m+6} + V_m A_{m+8} = 0$$  \hspace{1cm} (8)

where

$$P_m = <m - 6|H|m>$$  \hspace{1cm} (9)

$$Q_m = <m - 4|H|m>$$  \hspace{1cm} (10)

$$R_m = <m - 2|H|m>$$  \hspace{1cm} (11)

$$S_m = <m|H|m>$$  \hspace{1cm} (12)

$$T_m = R_{m+2}$$  \hspace{1cm} (13)

$$U_m = Q_{m+4}$$  \hspace{1cm} (14)

$$V_m = P_{m+4}$$  \hspace{1cm} (15)

The eigen values calculated using this relation using matrix diagonalisation method \[8,10,11\] are tabulated in table-1.
Tale-1: Eigenvalues of New Operator and Comparison.

<table>
<thead>
<tr>
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<th>$h = p^2 - 100x^2 + x^4$ [7]</th>
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<td>-2485.867 880 343</td>
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<td>3</td>
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</table>

III. Conclusion

In one-dimensional general oscillator considered above we notice that hermitian operator has an equivalent operator whose eigenspectra remain invariant. Further we plot the $|\Psi_{N=0-3}|^2$ corresponding to Hamiltonian

$$H = x^2 - 2p^2 - 2p^4 + p^6$$

in fig-1. Similarly we plot the $|\Phi_{N=0-3}|^2$ corresponding to Hamiltonian

$$h = p^2 - 2x^2 - 2x^4 + x^6$$

in fig-2. From the figs it is clear that eventhough two systems are iso-spectral in nature but different from each other.

References


Figure 1: \( H = x^2 - 2p^2 - 2x^4 + x^6 \)

: \( |\Psi_{n=0-3}|^2 \) of Equivalent Sextic Oscillator
Figure 2: $h = p^2 - 2x^2 - 2x^4 + x^6$

$|\Phi_{n=0-3}|^2$ of Sextic Well Oscillator