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ON THE STABILITY PROBLEM OF QUADRATIC FUNCTIONAL EQUATIONS IN 2-BANACH SPACES

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ABSTRACT

In this paper, we prove the stability problem in the spirit of Hyers-Ulam, Rassias and Găvruta for the quadratic functional equation:

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+4f(x)-2f(y)$$

in 2-Banach spaces. These results extend the generalized Hyers-Ulam stability results by the quadratic functional equation in normed spaces to 2-Banach spaces.

Keywords

Stability, Quadratic functional equations, 2-Banach spaces

SUBJECT CLASSIFICATION

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1 INTRODUCTION

In 1940, Ulam [17] suggested the stability problem of functional equations concerning the stability of group homomorphisms as follows: When is it true that a mapping satisfying a stability problem of functional equation approximately must be close to an exact solution of the given functional equation? The famous Ulam stability problem was partially solved by Hyers [10] of Banach spaces in 1941. Th.M. Rassias [14] given a generalization of Hyers' theorem by proving the existence of unique linear mappings near approximate additive mappings. The paper of Rassias has provided a lot of influence in the development of what we call the generalized Hyers-Ulam-Rassias stability of functional equations. In 1982, J.M. Rassias [13] provided a generalizations of the Hyer-Ulam stability theorem which allows the Cauchy difference controlled by a product of different powers of norm. And then, the result of Th.M. Rassias theorem has been generalized by Găvruta [8] by replacing the unbounded Cauchy difference by a generalized control function.

The quadratic function the mapping $f(x)=cx^2$ satisfies the following functional equation

$$f(x+y)+f(x-y)=2f(x)+f(y) \tag{1.1}$$

and therefore the functional equation (1.1) is called the quadratic functional equation. The generalized Hyers-Ulam-Rassias stability theorem for the quadratic functional equation (1.1) was proved by [4, 5, 9, 16] and the references therein. Recently, many mathematicians came out with results in linear 2-normed spaces, analogous with that in classical normed spaces and Banach spaces. In particular, the stability problems of the functional equations in 2-normed spaces have investigated by [1, 3, 11,12]. Before we present our results, we introduce some basic facts concerning linear 2-normed spaces. The theory of linear 2-normed spaces was first developed by Gähler [7] in the mid 1960's, while that of 2-Banach spaces was studied later by White [18]. For further details on linear 2-normed spaces, refer to the books [2] and [6].

Definition 1.1. [7] Let X be a linear space over $\mathbb{R}$ with dim $X>1$ and let $||\cdot,\cdot||_2 : \mathrm{X}\times\mathrm{X}\to\mathbb{R}$ be a real-valued function. Then $(X,||\cdot,\cdot||_2)$ is called a linear 2-normed space if

(2N1) $||x,y||_2 > 0$ and $||x,y||_2=0$ if and only if x and y are linearly dependent;

(2N2) $||x,y||_2=||y,x||_2$;

(2N3) $||\alpha x,y||_2=|\alpha|\,||x,y||_2$ for all $\alpha\in\mathbb{R}$;

(2N4) $||x+y,z||_2\le||x,z||_2+||y,z||_2$

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for all $x, y, z \in X$. The function $||\cdot,\cdot||_2$ is called a 2-norm on X.

Geometrically, a 2-norm function generalizes the concept of area function of parallelogram due to the fact that it represents the area of the usual parallelogram spanned by the two associated elements. For example, take $X = \mathbb{R}^2$ being equipped with the 2-norm

$$||x, y||_2 = \text{the area of the parallelogram spanned by the vectors } x \text{ and } y,$$

which may be given explicitly by the formula

$$||x, y||_2 = |x_1 y_2 - x_2 y_1|$$

where $x = (x_1, y_1),\ y = (x_2, y_2)$. Then, $(X, ||\cdot,\cdot||_2)$ is a linear 2-normed space.

Remark 1.2. Let X be a linear 2-normed space.

(1) $||x, y + \alpha x||_2 = ||x, y||_2$ for all $\alpha \in \mathbb{R}$.

(2) If x, y and z are linearly dependent, then

$$||x, y + z||_2 = ||x, y||_2 + ||x, z||_2 \quad \text{or} \quad ||x, y - z||_2 = ||x, y||_2 + ||x, z||_2.$$

(3) It follows from the conditions (2N2) and (2N4) that

$$\left| ||x, z||_2 - ||y, z||_2 \right| \le ||x - y, z||_2$$

for all $x, y, z \in X$. Hence, the function $x \mapsto ||x, z||_2$ is continuous function of X into $\mathbb{R}$ for any fixed $z \in X$**.**

Lemma 1.3. Let X be a linear 2-normed space. If $x \in X$ and $||x, z||_2 = 0$ for all $z \in X$, then $x = 0$.

Proof. Suppose that $x \neq 0$. If $||x, z||_2 = 0||x, z||_2 \neq 0$ for all $z \in X$, then x, z are linearly dependent for all $z \in X$. Since the dim $X > 1$, the only way that x can be linearly dependent for all $z \in X$ is for $x = 0$.

Definition 1.4. [18] Let X be a linear 2-normed space.

(1) A sequence $\{x_n\}$ in X is called a 2-convergent sequence if there is an $x \in X$ such that $\lim_{n\to\infty} ||x_n - x, y||_2 = 0$ for all $y \in X$. If the sequence $\{x_n\}$ 2-converges to x, then we write $\lim_{n\to\infty} x_n = x$ and x is called the limit of $\{x_n\}$.

(2) A sequence $\{x_n\}$ in X is called a 2-Cauchy sequence if there are $y, z \in X$ such that y and z are linearly independent,

$$\lim_{m,n\to\infty} ||x_n - x_m, y||_2 = 0 \quad \text{and} \quad \lim_{m,n\to\infty} ||x_n - x_m, z||_2 = 0.$$

(3) A linear 2-normed space in which every 2-Cauchy sequence is a convergent sequence is called a 2-Banach space.

Example 1.5. Let $X = \mathbb{R}^3$ with the vector addition and scalar multiplication defined componentwise and with 2-norm defined as follows:

$$||x, y||_2 = |x \times y| = \text{abs}\begin{vmatrix} i & j & k \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

for all $x = a_1 i + b_1 j + c_1 k$ and $y = a_2 i + b_2 j + c_2 k$ in X. Then, $\left(X, ||\cdot,\cdot||_2\right)$ is a 2-Banach space.

Note that [2] (1) for a convergent sequence $\{x_n\}$ in a linear 2-normed space X, $\lim_{n\to\infty} ||x_n, y||_2 = ||\lim_{n\to\infty} x_n, y||_2$ for all $y \in X$,

(2) a sequence $\{x_n\}$ is a 2-Cauchy sequence in a linear 2-normed space X if and only if $\lim_{n,m\to\infty} ||x_n - x_m, y||_2 = 0$ for all $y \in X$.

In [15], Ravi et al. considered the following functional equation

$$f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 4f(x) - 2f(y) \tag{1.2}$$

Then, it is easy to show the function $f(x) = x^2$ satisfies the functional equation (1.2), which is called the quadratic functional equation and every solution of the quadratic functional equation is said to be a quadratic mapping. The functional equation (1.2) are equivalence to the standard Euler-Lagrange functional equation (1.1). In this paper, we investigate the stability problem in the spirit of Hyers-Ulam, Rassias and Găvruta for the quadratic functional equation (1.2) in 2-Banach spaces. These results extend the generalized Hyers-Ulam stability results by the quadratic functional equation in normed spaces to 2-Banach spaces.

Throughout this paper, let X be a real normed space. We consider that there is a 2-norm on X which makes $\left(X, ||\cdot,\cdot||_2\right)$ a 2-Banach space (in shortly, we substitute X_2 for $(X, ||\cdot,\cdot||_2)$). For any $x_1, x_2, \dots, x_n$ in X, let $V(x_1, x_2, \dots, x_n)$ denote the subspace of X generated by $x_1, x_2, \dots, x_n$. Whenever the notation $V(x_1, x_2, \dots, x_n)$ is used, the vectors $x_1, x_2, \dots, x_n$ are linearly independent. For a given mapping $f : X \to X_2$ we define a difference operator $Df: X \times X \to X_2$ by

$$Df(x,y) = f(2x+y) + f(2x-y) - 2f(x+y) - 2f(x-y) - 4f(x) + 2f(y)$$

for all $x, y \in X$.

2. MAIN RESULTS

2.1 Rassias stability of (1.2)

In this section, we give the generalized Hyers-Ulam-Rassias stability for the quadratic functional equation (1.2) in linear 2-normed spaces. The results obtained here extend the ones obtained by Hyers-Ulam [10], Th.M. Rassias [14] and J.M. Rassias [13].

Theorem 2.1. Let $\ell \in \{-1, 1\}$ be fixed, θ and p be positive real numbers with $\ell p < l$. If a mapping $f : X \to X_2$ satisfies the inequality

$$||Df(x,y), z||_2 \leq \theta ||x||^p ||y||^p ||z||^p \tag{2.1}$$

for all $x, y, z \in X$ with $z \notin V(x,y)$, then the limit $Q(x) = \lim_{n\to\infty} \frac{f(3^{\ell n}x)}{9^{\ell n}}$ exists for all $x \in X$ and the unique quadratic mapping $Q : X \to X_2$ satisfies (1.2) and the inequality

$$\|f(x) - Q(x), z\|_2 \leq \frac{\theta ||x||^{2P} ||z||^p}{\ell(9 - 3^{3p-1})} \tag{2.2}$$

for all $x, z \in X$ with $z \notin V(x)$.

Proof. Letting $x = y$ in (2.1), we have

$$||f(3x) - 3f(x) - 2f(2x) - 2f(0), z||_2 \leq \theta ||x||^{2P} ||z||^p$$

for all $x, z \in X$ with $z \notin V(x)$ and replacing y by 0 in (2.1), we have

$$||2f(2x) - 8f(x) - 2f(0), z||_2 \leq 0$$

for all $x, z \in X$ with $z \notin V(x)$. Then, it follows from the above inequalities that

$$\left\| f(x) - \frac{f(3x)}{9}, z \right\|_2 \leq \frac{\theta ||x||^{2P} ||z||^p}{9}$$

for all $x, z \in X$ with $z \notin V(x)$. Replacing (x, z) by $(3^{\ell j}x, 3^{\ell j}z)$, we have

$$\left\| \frac{f(3^{\ell j}x)}{9^{\ell j}} - \frac{f(3^{\ell(j+1)}x)}{9^{\ell(j+1)}}, z \right\|_2 \leq \frac{\theta}{9} \left(\frac{3^p}{3}\right)^{3\ell j} ||x||^{2p} ||z||^p$$

for all $x, z \in X$ with $z \notin V(x)$ and all integers $j \in \mathbb{Z}$ with $j \geq 0$. Thus, we can obtain

$$\left\| \frac{f(3^{\ell k}x)}{9^{\ell k}} - \frac{f(3^{\ell m}x)}{9^{\ell m}}, z \right\|_2 \leq \frac{\theta ||x||^{2P} ||z||^p}{9} \sum_{j=k+\frac{1-\ell}{2}}^{m-\frac{1+\ell}{2}} \left(\frac{3^p}{3}\right)^{3\ell j} \tag{2.3}$$

for all $x, z \in X$ with $z \notin V(x)$ and all integers $k, m \in \mathbb{Z}$ with $m > k \geq 0$. Since $\ell p < p$,

$$\lim_{k,m\to\infty} \left\| \frac{f(3^{\ell k}x)}{9^{\ell k}} - \frac{f(3^{\ell m}x)}{9^{\ell m}}, z \right\|_2 = 0$$

for all $x, z \in X$ with $z \notin V(x)$. This means that the sequence $\left\{\frac{f(3^{\ell n}x)}{9^{\ell n}}\right\}$ is a 2-Cauchy sequence in a 2-Banach space X_2, the sequence $\left\{\frac{f(3^{\ell n}x)}{9^{\ell n}}\right\}$ 2-converges for all $x \in X$. So, we can define a mapping $Q : X \to X_2$ by

$$Q(x) = \lim_{n\to\infty} \frac{f(3^{\ell n}x)}{9^{\ell n}}$$

for all $x \in X$. Taking the limit as $m \to \infty$ in (2.3) with $k = 0$, it is easy to see that we find the mapping Q satisfies (2.2). If we replace $(3^{\ell j}x, 3^{\ell j}y, 3^{\ell j}z)$ in the place of (x, y, z) and divide $3^{3\ell j}$ both sides of (2.1), we have

$$||DQ(x,y),z||_2 = \lim_{n\to\infty} \frac{1}{3^{3\ell n}} \left||Df(3^{\ell n}x, 3^{\ell n}y), 3^{\ell j}z\right||_2 \leq \lim_{n\to\infty} \theta||x||^p||y||^p||z||^p 3^{3(p-1)\ell n} = 0$$

for all $x, y, z \in X$ with $z \notin V(x, y)$. It follows from Lemma 1.3 that $DQ(x, y) = 0$ for all $z \in X$, Thus, the mapping Q is quadratic. To prove the uniqueness of the quadratic mapping Q, let us assume that there exists another quadratic mapping $Q' : X \to X_2$ which satisfies (1.2) and (2.2). Since we have $Q'(3^{\ell n}x) = 9^{\ell n}Q'(x)$ for all $x \in X$ and all integers $n \in \mathbb{Z}$ with $n \geq 0$, it follows from (2.2) that

$$\left\|\frac{f(3^{\ell n}x)}{9^{\ell n}} - Q'(x), z\right\|_2 \leq \frac{\theta||x||^{2P}||z||^p}{\ell(9 - 3^{3p-1})}\left(\frac{3^p}{3}\right)^{3\ell n}$$

for all $x, z \in X$ with $z \notin V(x)$, which tends to zero as $n \to \infty$ for all $x \in X$. So, we can conclude that $Q(x) = Q'(x)$ for all $x \in X$. This completes the proof. ■

Corollary 2.2. Let $\ell \in \{-1, 1\}$ be fixed, θ and p, q, r be positive real numbers with $\ell(p + q + r) < l$. If a mapping $f : X \to X_2$ satisfies the inequality

$$||Df(x,y),z||_2 \leq \theta||x||^p||y||^q||z||^r$$

for all $x, y, z \in X$ with $z \notin V(x, y)$, then the unique quadratic mapping $Q : X \to X_2$ satisfies (1.2) and the inequality

$$||\mathrm{f(x)} - \mathrm{Q(x)}, z||_2 \leq \frac{\theta}{\ell(9 - 3^{p+q+r-3})}||x||^{p+q}||z||^r$$

for all $x, z \in X$ with $z \notin V(x)$.

Corollary 2.3. Let δ be a positive real number and $f : X \to X_2$ be a mapping that satisfies the inequality

$$||Df(x,y),z||_2 \leq \delta$$

for all $x, y, z \in X$ with $z \notin V(x, y)$. Then there exists a unique quadratic mapping $Q : X \to X_2$ such that the inequality

$$||\mathrm{f(x)} - \mathrm{Q(x)}, z||_2 \leq \frac{3}{13}\delta$$

for all $x, z \in X$ with $z \notin V(x)$.

Now, we are going to establish the modified the Hyers-Ulam stability in the spirit of J.M. Rassias [13] which we replaced the factor $||x||^p + ||y||^p + ||z||^p$ by $||x||^p||y||^p||z||^p$ for the quadratic functional equation (1.2).

Theorem 2.4. Let $\ell \in \{-1, 1\}$ be fixed, θ and p be positive real numbers with $\ell p < 3\ell$. If a mapping $f : X \to X_2$ satisfies the inequality

$$||Df(x,y),z||_2 \leq \theta(||x||^p + ||y||^p + ||z||^p) \tag{2.4}$$

for all $x, y, z \in X$ with $z \notin V(x, y)$, then the limit $Q(x) = \lim_{n\to\infty} \frac{f(3^{\ell n}x)}{9^{\ell n}}$ exists for all $x \in X$ and the unique quadratic mapping $Q : X \to X_2$ satisfies (1.2) and the inequality

$$||f(x) - Q(x), z||_2 \leq \frac{\theta||x||^p}{\ell(3-3^{p-2})} + \frac{2\theta||z||^p}{\ell(9-3^{3p-1})} \tag{2.5}$$

for all $x, z \in X$ with $z \notin V(x)$.

Proof. Using the same method as in the proof of Theorem 2.1, we have

$$\left\|f(x) - \frac{f(3x)}{9}, z\right\|_2 \leq \frac{\theta}{9}(3||x||^P + 2||z||^p)$$

for all $x, z \in X$ with $z \notin V(x)$, and then we have

$$\left\|\frac{f(3^{\ell k}x)}{9^{\ell k}} - \frac{f(3^{\ell m}x)}{9^{\ell m}}, z\right\|_2 \leq \frac{\theta}{9}\sum_{j=k+\frac{1-\ell}{2}}^{m-\frac{1+\ell}{2}}\left(\frac{1}{3}\right)^{3\ell j}(3^{1+\ell jp}||x||^p + 2\cdot 3^{\ell jp}||z||^p) \tag{2.6}$$

for all $x, z \in X$ with $z \notin V(x)$ and all integers $k,\ m$ with $m > k \geq 0$. Then, the sequence $\left\{\frac{f(3^{\ell n}x)}{9^{\ell n}}\right\}$ is a 2-Cauchy sequence in the 2-Banach space X_2, and so the sequence $\left\{\frac{f(3^{\ell n}x)}{9^{\ell n}}\right\}$ 2-converges for all $x \in X$. We can define a mapping $Q : X \to X_2$ by $Q(x) = \lim_{n\to\infty} \frac{f(3^{\ell n}x)}{9^{\ell n}}$ for all $x \in X$. Letting $m \to \infty$ in (2.5) with $k = 0$,

$$\lim_{m\to\infty} \left\| f(x) - \frac{f(3^{\ell m}x)}{9^{\ell m}}, z \right\|_2 \leq \frac{\theta||x||^p}{\ell(3-3^{p-2})} + \frac{2\theta||z||^p}{\ell(9-3^{3p-1})}$$

for all $x, z \in X$. Thus, we obtain the desired inequality (2.5). The remaining proof is similar to that of the proof of Theorem 2.1. This completes the proof. ■

Theorem 2.5. Let $\ell \in \{-1, 1\}$ be fixed, $\theta > 0$ and p, q be positive real numbers with $\ell(p+q) < 3\ell$. If a mapping $f : X \to X_2$ satisfies the inequality

$$||Df(x,y), z||_2 \leq \theta(||x||^p + ||y||^p)\, ||z||^q \tag{2.7}$$

for all $x, y, z \in X$ with $z \notin V(x, y)$, then there exists a unique quadratic mapping $Q : X \to X_2$ satisfies (1.2) and the inequality

$$||f(x) - Q(x), z||_2 \leq \frac{\theta||x||^p||z||^q}{3\ \ell(1-3^{p+q-3})} \tag{2.8}$$

for all $x, z \in X$ with $z \notin V(x)$.

Proof. Letting $x = y$ in (2.7), we have

$$\|f(3x) - 3f(x) - 2f(2x) - 2f(0), z\|_2 \leq 2\theta||x||^P||z||^q$$

for all $x, z \in X$ with $z \notin V(x)$. Putting y by 0 in (2.7), we have

$$\|2f(2x) - 8f(x) - 2f(0), z\|_2 \leq \theta||x||^P||z||^q$$

for all $x, z \in X$ with $z \notin V(x)$. Then we obtain

$$\left\| \frac{f(3^{\ell k}x)}{9^{\ell k}} - \frac{f(3^{\ell m}x)}{9^{\ell m}}, z \right\|_2 \leq \frac{\theta||x||^P\, ||z||^q}{3} \sum_{j=k+\frac{1-\ell}{2}}^{m-\frac{1+\ell}{2}} 3^{\ell(p+q-3)j}$$

for all $x, z \in X$ with $z \notin V(x)$ and all integers $k,\ m$ with $m > k \geq 0$. Then, the sequence $\left\{\frac{f(3^{\ell n}x)}{9^{\ell n}}\right\}$ is a 2-Cauchy sequence in the 2-Banach space X_2, and so the sequence $\left\{\frac{f(3^{\ell n}x)}{9^{\ell n}}\right\}$ 2-converges in X. So, we may define a mapping $Q : X \to X_2$ by $Q(x) = \lim_{n\to\infty} \frac{f(3^{\ell n}x)}{9^{\ell n}}$ for all $x \in X$. Thus, the mapping Q is quadratic and satisfies (1.2) and (2.8). The remaining assertion goes through by similar method to be the proof Theorem 2.3. This completes the proof. ■

Now, we will provide an example to illustrate that the functional equation (1.2) is not stable for $p + q = 3$ is singular in Theorem 2.5:

Example 2.6. $X = \mathbb{C}$ be a linear space over $\mathbb{R}$. Define a mapping $||\cdot,\cdot||_2 : X \times X \to \mathbb{R}$ by

$$||x, y||_2 = |ad - bc|,$$

where $x = a + bi,\ y = c + di$ for $a, b, c, d \in \mathbb{R}$ ($i = \sqrt{-1}$ is the imaginary unit). Then $(X, ||\cdot,\cdot||_2)$ is a linear 2-normed space. Let $\phi: \mathbb{C} \to \mathbb{C}$ be defined by

$$\phi(x) = \begin{cases} x^2, & |x| < 1; \\ 1, & |x| \geq 1. \end{cases}$$

Consider the function $f : \mathbb{C} \to \mathbb{C}$ by the formula $f(x) = \sum_{j=0}^{\infty} \frac{1}{9^j}\, \phi(3^j x)$ for all $x \in \mathbb{C}$. It is clear that f is bounded by $\frac{9}{8}$ on $\mathbb{C}$. We prove that

$$||Df(x,y), z||_2 \leq 144(|x|^2 + |y|^2)|z|^2 \tag{2.9}$$

for all $x, y, z \in \mathbb{C}$ with $z \notin V(x, y)$. To see this, if $|x|^2 + |y|^2 = 0$ or $|x|^2 + |y|^2 \geq \frac{1}{9}$, then we have (2.9). Now, suppose that $0 < |x|^2 + |y|^2 < \frac{1}{9}$. Then, there exists a positive integer $k \in \mathbb{Z}^+$ such that

$$\frac{1}{9^{k+1}} < |x|^2 + |y|^2 < \frac{1}{9^k}. \tag{2.10}$$

Then, $9^{k-1}\,|x|^2 < \frac{1}{9}$, $9^{k-1}\,|y|^2 < \frac{1}{9}$ and $3^{k-1}(2x \pm y)$, $3^{k-1}(x \pm y), 3^{k-1}x, 3^{k-1}y \in (-1,1)$. For $j = 0,1,\dots,k-1$,

$$\phi\left(3^j(2x+3)\right) + \phi\left(3^j(2x-y)\right) - 2\phi\left(3^j(x+y)\right) - 2\phi\left(3^j(x-y)\right)$$
$$-4\phi\left(3^j(x)\right) + 2\phi\left(3^j(y)\right) = 0.$$

It follows from the definition of f and (2.10) that

$$||Df(x,y),z||_2 \le 12\sum_{j=0}^{\infty}\frac{1}{9^j}|z| = \frac{27}{2}\frac{1}{9^k}|z| \le 144(|x|^2+|y|^2)|z|^2.$$

Thus, f satisfies (2.9) for all $x, y, z \in \mathbb{R}$ with $0 < |x|^2 + |y|^2 < \frac{1}{9}$. Assume that there exist a quadratic mapping $Q : \mathbb{C} \to \mathbb{C}$ and a positive constant $\mu \in \mathbb{R}^+$ such that

$$||f(x) - Q(x), z||_2 \le \mu|x|^2|z|$$

for all $x, z \in \mathbb{C}$ with $z \notin V(x)$. Then there exists a constant $c \in \mathbb{C}$ such that $Q(x) = cx^2$ for all rational numbers $x \in \mathbb{Q}$. So we have

$$||f(x), z||_2 \le (\mu + |c|)|x|^2|z| \tag{2.11}$$

for all $x \in \mathbb{Q}$ and all $z \in \mathbb{C}$. Let $\lambda \in \mathbb{Z}^+$ with $\lambda > \mu + |c|$. If $z = si$ where $s \in \mathbb{R}$ and x is a rational number in $(0, \frac{1}{3^\lambda})$, then $3^j x \in (0,1)$ for all $j = 0,1,\dots,\lambda-1$, and we obtain

$$||Df(x,y),z||_2 = \left\|\sum_{j=0}^{\infty}\frac{1}{9^j}\,\phi(3^j x), z\right\|_2 \ge \sum_{j=0}^{\lambda-1}\frac{1}{9^j}\,\phi(3^j x)|s| = \lambda|x|^2|s| > (\mu+|c|)|x|^2|z|$$

which contradicts (2.11). Therefore the quadratic functional equation (1.2) is not stable in sense of Ulam, Hyers, Rassias if $p + r = 3$.

2.2 Găvrute stability of (1.2) ăăă

In this section, we obtain stability in the spirit of Găvrute [8] for the quadratic functional equation (1.2). This results generalized and modifies the Hyers-Ulam stability in 2-Banach spaces.

Theorem 3.1. Let $\ell \in \{-1, 1\}$ be fixed, $\psi : X^3 \to \mathbb{R}$ be a mapping such that

$$\hat{\psi}(x,z) = \sum_{n=\frac{1-\ell}{2}}^{\infty}\frac{1}{3^{3\ell n}}\left(\psi(3^{\ell n}x, 3^{\ell n}x, 3^{\ell n}z) + \psi(3^{\ell n}x, 0, 3^{\ell n}z)\right) < \infty$$

and $\lim_{n\to\infty}\frac{1}{9^{\ell n}}\psi(3^{\ell n}x, 3^{\ell n}y, z) = 0$ for all $x, y, z \in X$. If a mapping $f : X \to X_2$ satisfies the inequality

$$||Df(x,y),z||_2 \le \psi(x,y,z) \tag{3.1}$$

for all $x, y, z \in X$ with $z \notin V(x,y)$, then the limit $Q(x) = \lim_{n\to\infty}\frac{f(3^{\ell n}x)}{9^{\ell n}}$ exists for all $x \in X$ and the unique quadratic mapping $Q : X \to X_2$ satisfies (1.2) and the inequality

$$||f(x) - Q(x), z||_2 \le \frac{1}{9}\hat{\psi}(x,z) \tag{3.2}$$

for all $x, z \in X$ with $z \notin V(x)$.

Proof. Letting $x = y$ in (3.1), we have

$$||f(3x) - f(x) - 2f(2x) - 2f(0), z||_2 \le \psi(x,x,z)$$

for all $x, z \in X$ with $z \notin V(x)$. Putting $y = 0$ in (3.1), we have

$$\left\|2f(2x)-3f(x)+2f(0),z\right\|_2 \le \psi(x,0,z)$$

for all $x,z\in X$ with $z\notin V(x)$. Then it follows from the above inequalities that

$$\left\|\left\|f(x)-\frac{1}{9}f(3x),z\right\|\right\|_2 \le \frac{1}{9}\big(\psi(x,x,z)+\psi(x,0,z)\big)$$

for all $x,z\in X$ with $z\notin V(x)$. Thus, we have

$$\left\|\frac{f(3^{\ell j}x)}{9^{\ell j}}-\frac{f(3^{\ell(j+1)}x)}{9^{\ell(j+1)}},z\right\|_2 \le \left(\frac{1}{3}\right)^{2+3\ell j}\left(\psi(3^{\ell j}x,3^{\ell j}x,3^{\ell j}z)+\psi(3^{\ell j}x,0,3^{\ell j}z)\right)$$

for all $x,z\in X$ and all integers $j\in\mathbb{Z}$ with $j\ge 0$, which is

$$\left\|\frac{f(3^{\ell k}x)}{9^{\ell k}}-\frac{f(3^{\ell m}x)}{9^{\ell m}},z\right\|_2 \le \frac{1}{9}\sum_{j=k+\frac{1-\ell}{2}}^{m-\frac{1-\ell}{2}}\left(\frac{1}{3}\right)^{3\ell j}\left(\psi(3^{\ell j}x,3^{\ell j}x,3^{\ell j}z)+\psi(3^{\ell j}x,0,3^{\ell j}z)\right) \quad (3.3)$$

for all $x,z\in X$ with $z\notin V(x)$ and all integers $k,\ m\in\mathbb{Z}$ with $m>k\ge 0$. Then, the sequence $\left\{\frac{f(3^{\ell n}x)}{9^{\ell n}}\right\}$ is a 2-Cauchy sequence in the 2-Banach space X_2, and so it 2-converges for all $x\in X$. So, we may define a mapping $Q: X\to X_2$ by $Q(x)=\lim_{n\to\infty}\frac{f(3^{\ell n}x)}{9^{\ell n}}$ for all $x\in X$. Letting $m\to\infty$ in (3.3) with $k=0$, we obtain the desired inequality (3.2). The remaining proof is similar to that of the proof of Theorem 2.1. This completes the proof.

Remark 3.2. Let $\psi(x,y,z)=\theta\|x\|^p\|y\|^p\|z\|^p$ with $\ell p<\ell$ and $\psi(x,y,z)=\theta(\|x\|^p+\|y\|^p+\|z\|^p)$ with $\ell p<3\ell$ in Theorem 3.1, respectively. Then we obtain (2.2) in Theorem 2.1 and (2.5) in Theorem 2.3.

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