Generalized Hyers-Ulam stability of derivations on Lie $C^*$-algebras

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Abstract: In this paper, we investigate new generalized Hyers-Ulam stability results for $(\alpha, \beta, \gamma)$-derivations on Lie $C^*$-algebras associated with the following $(m, n)$-Cauchy-Jensen additive functional equation:

$$
\sum_{1 \leq i < j \leq n} f\left(\frac{1}{m} \sum_{j=1}^{m} x_{ij} + \sum_{k=1}^{n} x_{ki} \right) = \frac{n-m+1}{n} \left( \sum_{j=1}^{m} f(x_j) \right)
$$

by using the fixed point and direct methods.

Keywords: $(\alpha, \beta, \gamma)$-derivation, $(m, n)$-Cauchy-Jensen additive functional equation, Lie $C^*$-algebra, generalized Hyers-Ulam stability.
1 Introduction

The theory of finite dimensional complex Lie algebras is an important part of Lie theory. Lie algebras have many applications in physics and connections with other parts of mathematics. With an increasing amount of theory and applications concerning Lie algebras of various dimensions, it is becoming necessary to ascertain which tools are applicable for handling them. The miscellaneous characteristics of Lie algebras constitute such tools and have also found applications in Casimir operators [1], derived, lower central and upper central sequences, the Lie algebra of derivations, radicals, nilradicals, ideals, subalgebras [11], [20] and megaideals [19]. These characteristics are particularly crucial when considering possible affinities among Lie algebras. Recently, some authors have studied the stability problems of some functional equations in the setting of Lie algebras.

The stability problem concerning the stability of group homomorphisms of functional equations was originally introduced by Ulam [24], in 1940, as follows:

Let \((G_1,\ast)\) be a group and \((G_2,\circ,d)\) be a metric group with the metric \(d(\cdot,\cdot)\). Given \(\varepsilon > 0\), does there exist a \(\delta(\varepsilon) > 0\) such that if a mapping \(h: G_1 \rightarrow G_2\) satisfies the inequality \(d(h(x\ast y),h(x)\circ h(y)) < \varepsilon\) for all \(x,y \in G_1\), then there is a homomorphism \(H: G_1 \rightarrow G_2\) with \(d(h(x),H(x)) < \varepsilon\) for all \(x \in G_1\)?

If the answer is affirmative, we would say that the equation of a homomorphism \(H(x\ast y) = H(x)\circ H(y)\) is stable. The famous Ulam stability problem was partially solved by Hyers [10] for linear functional equation of Banach spaces. Later, the results of Hyers were generalized by Aoki [2], Găvruţa [8] and Rassias [23]. Cădariu and Radu [4] applied the fixed point method to investigation of the stability for a Jensen functional equation. They could present a short and a simple proof, which is different from the direct method initiated by Hyers in 1941, for the generalized Hyers-Ulam stability for a Jensen functional equation. In 2008, Novotný and Hrivnák [15] investigated generalizing the concept of Lie derivations via certain complex parameters and obtained various Lie and established the structure and properties of \((\alpha,\beta,\gamma)\)-derivations of Lie algebras.

A \(C^1\)-algebra \(A\) endowed with the Lie product \([x,y] = \frac{xy - yx}{2}\) on \(A\) is called a Lie \(C^1\)-algebra (see [16]). Let \(A\) be a \(C^1\)-algebra. A \(C\)-linear mapping \(D: A \rightarrow A\) is called a \((\alpha,\beta,\gamma)\)-derivation of \(A\) if there exist \(\alpha,\beta,\gamma \in C\) such that

\[
\alpha D([x,y]) = \beta [D(x), y] + \gamma [x, D(y)]
\]

for all \(x, y \in A\) (see [15]).

Let \(k\) be a fixed positive integer. We recall that a mapping \(\rho: A \rightarrow B\) having the domain \(A\) and the codomain \((B,\leq)\) that are both closed under addition is called a contractively subadditive mapping if there exists a constant \(L\) with \(0 < L < 1\) such that

\[
\rho(x + y) \leq L(\rho(x) + \rho(y))
\]

and an expansively superadditive mapping if there exists a constant \(L\) with \(0 < L < 1\) such that

\[
\rho(x + y) \geq \frac{1}{L}(\rho(x) + \rho(y))
\]

for all \(x, y \in A\). A mapping \(\rho: A \rightarrow B\) is called a homogeneous of degree \(k\) if

\[
\rho(\lambda x) = \lambda^k \rho(x)
\]

for all \(x \in A\). Also, if there exists a constant \(L\) with \(0 < L \leq 1\) such that a mapping \(\rho: A^n \rightarrow B\) satisfies

\[
\rho(x_1,\ldots,x_n,\lambda x_1,\ldots,x_n) \leq \lambda^{L} \rho(x_1,\ldots,x_n,\ldots,x_n)
\]

for all \(x_1,\ldots,x_n,x_1,\ldots,x_n \in A\) and positive integer \(\lambda\), then we say that \(\rho\) is a \(n\)-contractively subhomogeneous mapping if \(\ell = 1\) and \(\rho\) is an \(n\)-expansively superhomogeneous mapping if \(\ell = -1\). Note that \(\rho\) satisfies the properties...
\[
\rho(x_1, \ldots, x_n, \lambda^k x, x_{i_1}, \ldots, x_{i_m}, \ldots, x_n) \leq (\lambda^l L)^k \rho(x_1, \ldots, x_{i_1}, x, x_{i_1}, \ldots, x_n),
\]

Now, we consider a mapping \( f : X \to Y \) satisfying the following functional equation:

\[
\sum_{1 \leq i < \leq m, 1 \leq k \leq n} f\left(\frac{1}{m} \sum_{j=1}^{n} x_{i_j} + \sum_{k=1}^{n} x_{k_i}\right) = \frac{n-m+1}{n} \left(\sum_{j=1}^{n} f(x_j)\right).
\]

(1.1)

for all \( x_1, \ldots, x_n \in X \) where \( n, m \in \mathbb{Z}^+ \) are fixed integers with \( n \geq 2, 1 \leq m \leq n \).

We observe that, in case \( m = 1 \), the equation (1.1) yields the following Cauchy additive equation:

\[
\sum_{1 \leq i < \leq n} f(x_i) = n \sum_{i=1}^{n} f(x_i).
\]

Also, we observe that, in case \( m = n \), the equation (1.1) yields the following Jensen additive equation:

\[
\sum_{1 \leq i < \leq n} f\left(\frac{1}{n} \sum_{j=1}^{n} x_{i_j}\right) = \frac{1}{n} \sum_{j=1}^{n} f(x_j).
\]

Therefore, the functional equation (1.1) is a generalized form of the Cauchy-Jensen additive equation and thus every solution of the equation (1.1) may be analogously called the \( \text{general} \ (m, n) \text{-Cauchy-Jensen additive equation} \).

Let \( X \) and \( Y \) be linear spaces. For each \( m \in \mathbb{Z}^+ \) with \( 1 \leq m \leq n \), a mapping \( f : X \to Y \) satisfies the equation (1.1) for all \( n \geq 2 \) if and only if \( f(x) - f(0) = A(x) \) is Cauchy additive, where \( f(0) = 0 \) if \( m < n \). In particular, we have \( f((n-m+1)x) = (n-m+1)f(x) \) and \( f(mx) = mf(x) \) for all \( x \in X \).

Recently, Asgari et al. [3] established the generalized Hyers-Ulam-Rassias stability of the ternary homomorphisms and ternary derivations between fuzzy ternary Banach algebras associated to the functional equation (1.1). Rassias et al. [21] proved the generalized Hyers-Ulam stability of homomorphisms and derivations in \( C^* \)-ternary algebras associated with the functional equation (1.1). Also, Gordji et al. [9] investigated the stability of \( (\alpha, \beta, \gamma) \)-derivation on Lie \( C^* \)-algebras. For more details about the stability for various types of derivations, refer to [6], [12], [13], [14], [18] and [22].

In this paper, using some strategies from [9], [15] and [21], we investigate new stability of \( (\alpha, \beta, \gamma) \)-derivations on Lie \( C^* \)-algebras associated to the general \( (m, n) \)-Cauchy-Jensen type additive functional equation (1.1) by using the fixed point and direct methods.

Throughout this paper, let \( A \) be a Lie \( C^* \)-algebra, \( T_{h_{\theta}}^1 = \{ e^{i \theta} : 0 \leq \theta \leq 2 \pi / n \} \) and \( \lambda = n - m + 1 \in \mathbb{Z}^+ \) be a fixed positive integer with \( n \geq 2 \) and \( 1 \leq m \leq n \). For any mapping \( f : A \to A \), we define

\[
\Delta_{\mu} f(x_1, \ldots, x_n) = \sum_{1 \leq i < \leq m, 1 \leq k \leq n} f\left(\frac{1}{m} \sum_{j=1}^{n} \mu x_{i_j} + \sum_{k=1}^{n} \mu x_{k_i}\right) - \frac{n-m+1}{n} \left(\sum_{j=1}^{n} f(\mu x_j)\right).
\]

for all \( x_1, \ldots, x_n, x, y \in A, \ \mu \in T_{h_{\theta}}^1 \) and for some \( \alpha, \beta, \gamma \in \mathbb{C} \) .
2 Stability of \((\alpha, \beta, \gamma)\)-derivations on Lie \(C^*\)-algebras

In this section, we give some new generalized Hyers-Ulam stability results for \((\alpha, \beta, \gamma)\)-derivation on Lie \(C^*\)-algebras associated to the equation \(\Delta_n f(x_1, \ldots, x_n) = 0\) via two methods.

2.1. Fixed point method

Let us recall that a mapping \(d : X^2 \to [0, \infty)\) is called a generalized metric on a nonempty set \(X\) if (1) \(d(x, y) = 0\) if and only if \(x = y\); (2) \(d(x, y) = d(y, x)\); (3) \(d(x, z) \leq d(x, y) + d(y, z)\) for all \(x, y, z \in X\).

The following fixed point theorem proved by Diaz and Margolis [7] plays an important role in proving our theorem:

**Theorem 2.1** [7] Suppose that \((\Omega, d)\) is a complete generalized metric space and \(T : \Omega \to \Omega\) is a strictly contractive mapping with Lipshitz constant \(L\). Then, for each \(x \in \Omega\), either \(d(T^n x, T^{n+1} x) = \infty\) for all \(n \geq 0\) or there exists a natural number \(n_0\) such that

1. \(d(T^n x, T^{n+1} x) < \infty\) for all \(n \geq n_0\);
2. the sequence \(\{T^n x\}\) is convergent to a fixed point \(y^*\) of \(T\);
3. \(y^*\) is the unique fixed point of \(T\) in the set \(\Lambda = \{y \in \Omega : d(T^n x, y) < \infty\}\);
4. \(d(y, y^*) \leq \frac{1}{1-L} d(y, Ty)\) for all \(y \in \Lambda\).

**Theorem 2.2** Assume that there exist a contractively subadditive mapping \(\phi : A^\infty \to [0, \infty)\) and a 2-contractively subhomogeneous mapping \(\psi : A^2 \to [0, \infty)\) with the constant \(L < 1\) such that a mapping \(f : A \to A\) satisfies the following conditions:

\[
\left\| \Delta_n f(x_1, \ldots, x_n) \right\| \leq \phi(x_1, \ldots, x_n),
\]

\[
\left\| \alpha f([x, y]) - \beta f(x, y) - \gamma f(x, f(y)) \right\| \leq \psi(x, y)
\]

for all \(x_1, \ldots, x_n, x, y \in A\) and \(\mu \in T_{(n)}^{(m)}\) and for some \(\alpha, \beta, \gamma \in \mathbb{C}\). Then there exists a unique \((\alpha, \beta, \gamma)\)-derivation \(D : A \to A\) which satisfies the equation (1.1) and the following inequality:

\[
\left\| f(x) - D(x) \right\| \leq \frac{1}{n! (n-m+1)(1-L)} \phi(x, \ldots, x)
\]

for all \(x \in A\).

Proof. Letting \(x_1, \ldots, x_n = x\) and \(\mu = 1\) in (2.1), we have

\[
\left\| \sum_{m=1}^{n} \binom{n}{m} f((n-m+1)x) - \sum_{m=1}^{n} \binom{n}{m} (n-m+1)f(x) \right\| \leq \phi(x, \ldots, x),
\]

which gives
\[ \left\| f(x) - \frac{1}{\lambda} f(\lambda x) \right\| \leq \frac{1}{n^m} \phi(x, \ldots, x) \]  \tag{2.5}

for all \( x \in A \), where \( \lambda = n - m + 1 \).

Let \( \Omega \) be a set of all mappings from \( A \) into \( A \) and introduce a generalized metric on \( \Omega \) as follows:

\[ d(g, h) = \inf \{ C \in [0, \infty) : \left\| g(x) - h(x) \right\| \leq C \phi(x, \ldots, x) \text{ for all } x \in A \} \]

Then \((\Omega, d)\) is a generalized complete metric space ([5]). We consider the mapping \( T : \Omega \to \Omega \) defined by

\[ (Tg)(x) = \frac{1}{\lambda} g(\lambda x) \]

for all \( g \in \Omega \) and \( x \in A \). Let \( g, h \in \Omega \) and \( C \in [0, \infty) \) be an arbitrary constant with \( d(g, h) < C \).

Then we have

\[ \left\| (Tg)(x) - (Th)(x) \right\| \leq \frac{C}{\lambda} \phi(\lambda x, \ldots, \lambda x) \leq LC \phi(x, \ldots, x) \]  \tag{2.6}

for all \( x \in A \), which means that

\[ d(Tg, Th) \leq Ld(g, h) \]

for all \( g, h \in \Omega \). Then \( T \) is a strictly contractive self-mapping on \( \Omega \) with the Lipschitz constant \( L \). It follows from (2.5) that

\[ d(Tf, f) \leq \frac{1}{n^m} \lambda \]

Thus, from Theorem 2.1, there exists a mapping \( D \), which is a unique fixed point of \( T \) in the set \( \Omega_0 = \{ g \in \Omega : d(f, g) < \infty \} \), such that

\[ D(x) = \lim_{k \to \infty} \frac{1}{\lambda^k} f(\lambda^k x) = \lim_{k \to \infty} \frac{1}{(n - m + 1)^k} f((n - m + 1)^k x) \]  \tag{2.7}

for all \( x \in A \) since \( \lim_{k \to \infty} d(T^{k+1} f, \delta) = 0 \). Again, from Theorem 2.1, we have

\[ d(f, D) \leq \frac{1}{1 - \lambda} d(Tf, f) \leq \frac{1}{n^m \lambda (1 - \lambda)} = \frac{1}{n^m (n - m + 1)(1 - \lambda)} \]  \tag{2.8}

and so the inequality (2.3) holds. From (2.1), (2.7) and a contractive subadditive mapping of \( \phi \), it follows that

\[ \left\| \Delta \mu D(x_1, \ldots, x_n) \right\| \leq \lim_{k \to \infty} \left\| \Delta \mu f(\lambda^k x_1, \ldots, \lambda^k x_n) \right\| \leq \lim_{k \to \infty} \mu^k \phi(x_1, \ldots, x_n) = 0, \]  \tag{2.9}

which gives \( \Delta \mu D(x_1, \ldots, x_n) = 0 \) for all \( x_1, \ldots, x_n \in A \), \( \mu \in T_{\lambda}^{(1)} \). If we put \( \mu = 1 \) in (2.9), then \( D \) is additive. Also, letting \( x_1 = x \) and \( x_2 = \ldots = x_n = 0 \) in (2.9), we have \( D(\mu x) = \mu D(x) \). By the same reasoning as is the proof of Theorem 2.1 of [17], the mapping \( D \in \Omega \) is \( C \)-linear.
It follows from the linearity of $D$ and (2.2) that
\[
\left\| aD([x,y]) - \beta[D(x), y] - \gamma[x, D(y)] \right\|
\leq \lim_{k \to \infty} \left\| \frac{1}{\lambda^k} f(\lambda^k x, \lambda^k y) - \beta f(x, y) \right\|
\leq \lim_{k \to \infty} \frac{1}{\lambda^k} \psi(\lambda^k x, \lambda^k y) \leq \lim_{k \to \infty} \lambda^k \psi(x, y) = 0
\]
for all $x, y \in A$ and for some $\alpha, \beta, \gamma \in C$. Then we have
\[
\alpha D([x,y]) = \beta D(x,y) + \gamma [x,D(y)]
\]
for all $x, y \in A$ and for some $\alpha, \beta, \gamma \in C$. Therefore, $D$ is a unique $(\alpha, \beta, \gamma)$-derivation on a Lie $C^\ast$-algebra $A$ satisfying (2.3). This complete the proof.

**Theorem 2.3** Assume that there exist an expansively superadditive mapping $\phi : A^n \to [0, \infty)$ and a 2-expansively superhomogeneous mapping $\psi : A^2 \to [0, \infty)$ with the constant $L < 1$ such that a mapping $f : A \to A$ satisfies (2.1) and (2.2). Then there exists a unique $(\alpha, \beta, \gamma)$-derivation $D : A \to A$ which satisfies the equation (1.1) and the following inequality:
\[
\left\| f(x) - D(x) \right\| \leq \frac{L}{\left( \frac{n}{m} \right)} \phi(x, \ldots, x)
\]
for all $x \in A$.

Proof. It follows from (2.4) and an expansively superadditive mapping of $\phi$ that
\[
\left\| \lambda f(\frac{x}{\lambda}) - f(x) \right\| \leq \frac{L}{\left( \frac{n}{m} \right)} \phi(\frac{x}{\lambda}, \ldots, x)
\]
(2.11)
for all $x \in A$.

Let $\Omega$ and $d$ be same as in the proof of Theorem 2.2. Then $(\Omega, d)$ is a generalized complete metric space. We consider the mapping $T : \Omega \to \Omega$ defined by
\[
(Tg)(x) = \lambda g(\frac{x}{\lambda})
\]
for all $g \in \Omega$, $x \in A$. Thus we have $d(Tg, Th) \leq Ld(g, h)$ for all $g, h \in \Omega$ and, by (2.11),
\[
d(Tf, f) \leq \frac{L}{\left( \frac{n}{m} \right)} = \frac{L}{\left( \frac{n}{m} \right)} \left( \frac{n}{m} + 1 \right)
\]
It follows from Theorem 2.1 that there exists a mapping $D$, which is a unique fixed point of $T$ in the set $\Omega = \{ g \in \Omega : d(f, g) < \infty \}$, such that
for all \( x \in A \). Then we have
\[
d(f, D) \leq \frac{1}{1-L} d(D, f) \leq \frac{L}{n} \left( \frac{n}{m} \right) (1-L) = \frac{L}{n} \left( \frac{n}{m} (n-m+1) \right) (1-L)
\]
and so the inequality \((2.10)\) holds. The remaining assertion goes through in the similar method to the corresponding part of Theorem 2.2. This complete the proof.

From Theorems 2.2 and 2.3, we have the following:

**Corollary 2.4** Let \( r, s \in \mathbb{R} \) with \( r \neq 1, s \neq 2 \) and \( \theta \) be positive real numbers. Suppose that a mapping \( f : A \to A \) satisfies the following conditions:
\[
\left\| \Delta f(x_1, \ldots, x_n) \right\| \leq \theta \sum_{i=1}^{n} x_i^r, \quad (2.12)
\]
\[
\left\| \alpha f(x) - \beta f(y) - \gamma (x, f(y)) \right\| \leq \theta \left\| x \right\|^s + \left\| y \right\|^s \quad (2.13)
\]
for all \( x_1, \ldots, x_n, x, y \in A, \ \mu \in T_{\mathbb{R}} \) and for some \( \alpha, \beta, \gamma \in \mathbb{C} \). Then there exists a unique \( (\alpha, \beta, \gamma) \)-derivation \( D : A \to A \) which satisfies the equation \((1.1)\) and the following inequality:
\[
\left\| f(x) - D(x) \right\| \leq \frac{n \theta}{(n-m+1)} \left( \frac{n}{m} \right) (1-L) \left[ (n-m+1) - (n-m+1)^r \right], \quad r < 1, s < 2; \quad (2.14)
\]
for all \( x \in A \).

Proof. The proof follows from Theorems 2.2 and 2.3 by taking \( \phi(x_1, \ldots, x_n) = \theta \sum_{i=1}^{n} x_i^r \) and \( \psi(x, y) = \theta \left\| x \right\|^s + \left\| y \right\|^s \) for all \( x_1, \ldots, x_n, x, y \in A \). Then we can choose \( L = (n-m+1)^{1-r} \) if \( r < 1 \) and \( L = (n-m+1)^{s-r} \) if \( r > 2 \), respectively, and so we can obtain the desired result.

**Corollary 2.5** Let \( r, s \in \mathbb{R} \) with \( r = \sum_{i=1}^{n} r_i \neq 1 \), \( s \neq 2 \) and \( \theta \) be positive real numbers. Suppose that a mapping \( f : A \to A \) satisfies the following conditions:
\[
\left\| \Delta f(x_1, \ldots, x_n) \right\| \leq \theta \sum_{i=1}^{n} x_i^r, \quad (2.15)
\]
\[
\left\| \alpha f(x) - \beta f(y) - \gamma (x, f(y)) \right\| \leq \theta \left\| x \right\|^s + \left\| y \right\|^s \quad (2.16)
\]
for all $x_1, \ldots, x_n, x, y \in A$ and for some $\alpha, \beta, \gamma \in \mathbb{C}$. Then there exists a unique $(\alpha, \beta, \gamma)$-derivation $D : A \to A$ which satisfies the equation $A^{(1.1)}$ and the following inequality:

$$
\|f(x) - D(x)\| \leq \frac{\theta \|x\|}{\binom{n}{m}((n-m+1) - (n-m+1)'}, \quad r < 1, s < 2:
$$

$$
\|f(x) - D(x)\| \leq \frac{\theta \|x\|}{\binom{n}{m}((n-m+1)' - (n-m+1))}, \quad r > 2, s > 2:
$$

(2.17)

for all $x \in A$.

Proof. Putting $\phi(x_1, \ldots, x_n) = \theta \prod_{i=1}^n \|x_i\|^r$ and $\psi(x, y) = \theta \|x\|^r \|y\|^s$ for all $x_1, \ldots, x_n, x, y \in A$ and choosing $L = (n-m+1)^{-r}$ if $r < 1$ and $L = (n-m+1)^{1-r}$ if $r > 1$, respectively, we obtain the desired result by Theorems 2.2 and 2.3.

2.2. Direct method

In this subsection, we apply the direct method to investigate the new generalized Hyers-Ulam stability results for $(\alpha, \beta, \gamma)$-derivations on Lie $C^*$-algebras associated with the equation $A^{(1.1)}$.

**Theorem 2.6** Assume that there exist a contractively subadditive mapping $\phi : A^n \to [0, \infty)$ and a 2-contractively subhomogeneous mapping $\psi : A^2 \to [0, \infty)$ with the constant $L < 1$ such that a mapping $f : A \to A$ satisfies $A^{(2.1)}$ and $A^{(2.2)}$. Then there exists a unique $(\alpha, \beta, \gamma)$-derivation $D : A \to A$ which satisfies the equation $A^{(1.1)}$ and the following inequality:

$$
\|f(x) - D(x)\| \leq \frac{1}{\binom{n}{m}(n-m+1)(1-L)} \phi(x, \ldots, x)
$$

(2.18)

for all $x \in A$.

Proof. If we replace $x$ by $\lambda^j x$ and divide $\lambda^j$ both sides of $A^{(2.5)}$, then we have

$$
\|f(\lambda^j x) - f(\lambda^{j+1} x)\| \leq \frac{1}{\binom{n}{m}} \phi(\lambda^j x, \ldots, \lambda^j x)
$$

(2.19)

for all $x \in A$. Thus we have

$$
\|f(\lambda^j x) - f(\lambda^j x)\| \leq \frac{1}{\binom{n}{m}} \sum_{j=k}^{\infty} \frac{1}{\lambda^j} \phi(\lambda^j x, \ldots, \lambda^j x) \leq \frac{1}{\binom{n}{m}} \sum_{j=k}^{\infty} \lambda^j \phi(x, \ldots, x)
$$

(2.20)

for all $x \in A$ and $l, k \in \mathbb{Z}$ with $l > k \geq 0$, which implies that the sequence $\left\{ \frac{f(\lambda^j x)}{\lambda^j} \right\}$ is a Cauchy sequence.
sequence for all \( x \in A \) and so it converges. Thus we can define a mapping \( D : A \to A \) by
\[
D(x) = \lim_{\lambda \to \infty} \frac{f(\lambda x)}{\lambda}
\]
for all \( x \in A \). Then we have
\[
\left\| \Delta_{\mu} D(x_1, \ldots, x_n) \right\| \leq \lim_{\lambda \to \infty} \frac{1}{\lambda} \left\| \Delta_{\mu} f(\lambda x_1, \ldots, \lambda x_n) \right\|
\]
(2.20)
which gives \( \Delta_{\mu} D(x_1, \ldots, x_n) = 0 \) for all \( x_1, \ldots, x_n \in A \) and \( \mu \in T_{10}^1 \). If we put \( \mu = 1 \) in (2.20), then the mapping \( D \) is additive. Letting \( x_1 = x \) and \( x_2 = \cdots = x_n = 0 \), we have
\[
D(x, \ldots, x) = 0.
\]
By the same reasoning as is the proof of Theorem 2.1 of [17], \( D \) is linear. Further, we have
\[
\left\| \alpha D(x, y) - \beta D(x, y) - \gamma D(x, y) \right\|
\]
(2.19)
for all \( x, y \in A \). So, we have
\[
\alpha D(x, y) = \beta D(x, y) + \gamma D(x, y)
\]
for all \( x, y \in A \) and for some \( \alpha, \beta, \gamma \in C \). Letting \( l \to \infty \) in (2.19) with \( k = 0 \), we can find that the mapping \( D \) is a \( (\alpha, \beta, \gamma) \)-derivation on a Lie \( C^* \)-algebra \( A \) satisfying (2.18).

Next, let \( D' : A \to B \) be another \( (\alpha, \beta, \gamma) \)-derivation on \( A \) satisfying (2.18). Then we have
\[
\left\| D(x) - D'(x) \right\|
\]
(2.21)
for all \( x \in A \). Thus we can conclude that \( D(x) = D'(x) \) for all \( x \in A \). This complete the proof.

**Theorem 2.7** Assume that there exist an expansively superadditive mapping \( \phi : A^n \to [0, \infty) \) and a 2-expansively superhomogeneous mapping \( \psi : A^2 \to [0, \infty) \) with the constant \( L < 1 \) such that a mapping \( f : A \to A \) satisfies (2.1) and (2.2). Then there exists a unique \( (\alpha, \beta, \gamma) \)-derivation \( D : A \to A \) which satisfies the equation (1.1) and the following inequality:
\[
\left\| f(x) - D(x) \right\| \leq \frac{L}{\left( \begin{array}{c} n \\ m \end{array} \right)(n - m + 1)(1 - L) \right) \phi(x, \ldots, x)
\]
(2.21)
for all \( x \in A \).

Proof. The proof is similar to that of Theorem 2.6.
Corollary 2.8 Let $\theta_1, \theta_2$ be positive real numbers. Suppose that a mapping $f : A \to A$ satisfies
\begin{align*}
\| \Delta f(x_1, \ldots, x_n) \| &\leq \theta_1, \\
\| \alpha f([x, y]) - \beta f(x, y) - \gamma [x, f(y)] \| &\leq \theta_2
\end{align*}
for all $x_1, \ldots, x_n, y \in A$ and for some $\alpha, \beta, \gamma \in \mathbb{C}$. Then there exists a unique $(\alpha, \beta, \gamma)$-derivation $D : A \to A$ which satisfies
\begin{align*}
\| f(x) - D(x) \| &\leq \frac{\theta_1}{(n - m)},
\end{align*}
for all $x \in A$.

Remark. Let $\phi : A^n \to [0, \infty)$ and $\psi : A^2 \to [0, \infty)$ be mappings such that
\begin{align*}
\sum_{j=0}^{\infty} \frac{1}{n^{j+1}} \phi(\lambda x_1, \ldots, \lambda x_n) < \infty
\end{align*}
and
\begin{align*}
limit_{j \to \infty} \frac{1}{\lambda^j} \phi(\lambda x_1, \ldots, \lambda x_n) = 0, \quad limit_{j \to \infty} \frac{1}{\lambda^j} \psi(\lambda x, \lambda y) = 0
\end{align*}
for all $x_1, \ldots, x_n, x, y \in A$, where $\lambda = n - m + 1$. Suppose that $f : A \to A$ is a mapping which satisfies (2.1) and (2.2). By similar method to the proof Theorem 2.6, we can show that there exists a unique $(\alpha, \beta, \gamma)$-derivation $D : A \to A$ which satisfies (1.1) and
\begin{align*}
\| f(x) - D(x) \| &\leq \frac{1}{\lambda^j} \sum_{j=0}^{\infty} \frac{1}{n^{j+1}} \phi(\lambda x_1, \ldots, \lambda x_n)
\end{align*}
for all $x \in A$.

For the case $\phi(x_1, \ldots, x_n) = \epsilon + \theta \sum_j x_j$ and $\psi(x, y) = \epsilon + \theta \| x \| + \| y \|$ (where $\epsilon, \theta$ are positive real numbers and $r, s \in \mathbb{R}$ with $r < 1, s < 2$), there exists a unique $(\alpha, \beta, \gamma)$-derivation $D : A \to A$ satisfying
\begin{align*}
\| f(x) - D(x) \| &\leq \frac{\epsilon}{(n - m)} + \frac{n \theta \| x \|}{(n - m) ((n - m + 1) - (n - m + 1)^s)}
\end{align*}
for all $x \in A$. 
References