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A Parametric Approach for Solving Interval–Valued fractional Continuous Static Games

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Abstract

The aim of this paper is to show that a parametric approach can be used to solve fractional continuous static games with interval-valued in the objective function and in the constraints. In this game, cooperation among all the players is possible, and each player helps the others up to the point of disadvantage to himself, so we use the Pareto-minimal solution concept to solve this type of game. The Dinkelbach method is used to transform fractional continuous static games into non- fractional continuous static games. Moreover, an algorithm with the corresponding flowchart to explain the suggested approach is introduced. Finally, a numerical example to illustrate the algorithm's steps is given.

Keywords: Continuous Static Games, Nonlinear Programming Problem, Fractional Programming Problem, Interval-Valued Optimization.

1. Introduction

The game appears when there exists the more case of multiple decision-makers, each their own cost criterion. This generalization introduces the possibility of competition among the system controllers, called "players and the optimization problem under consideration is therefore termed a 'game." Each player in the game controls a specified subset of the system parameters (called his control vector) and seeks to minimize his own scalar cost criterion subject to specified constraints.

In [1] Thomas and Walter presented formulations and the solution of different types of continuous static games and differential games. The major classes of games are matrix games, continuous static games, and differential game.

In continuous static games **(CSG),** the decisions and costs are related in continuous rather than a discrete manner. The game is static in the sense that no time history is involved in the relationship between costs and decisions.

Applications of the game theory may be found in economics, engineering, and biology. There are several solution concepts are possible such as the Pareto-minimal concept, the Nash equilibrium concept, the min-max concept, and the Stackelberg leader-follower concept to solve **CSG** [1,2].

The fractional programming **(FP)** problem is considered a special case of nonlinear programming, which is generally used for modeling real-life problems with one or more objectives such as profit/cost. The **FP** is applied to different disciplines such as engineering, business, finance, economics, health care, and hospital planning. In the recent years, we have seen many approaches to solve fractional programming problem [3,4,5,6,7,8,9,10].

Interval-valued optimization problems may provide an alternative choice for considering the uncertainty into the optimization problems [11,12]. That is to say that the coefficients in the interval-valued optimization problems are assumed as closed intervals.

In this paper, we introduce a solution method to solve fractional continuous static games problem with interval-valued parameters in both the objective function and in both the right hand said and left hand said of constrains.

Also, an algorithm and a numerical example to illustrate the algorithm's steps are given.

2. Interval Analysis [11,12]

Let us denote by I the set of all closed and bounded interval in R. suppose, $A = [a^l, a^u]$ and $B = [b^l, b^u]$, we have the following operation on I.

1. $A + B = \{a + b | a\epsilon A \text{ and } b\epsilon B\} = [a^l + b^l, a^u + b^u] \in I.$

2. $-A = \{-a : a\epsilon A\} = [-a^u, -a^l].$

3. $A - B = [a^l - b^u, a^u - b^l]\epsilon I.$

4. $kA = \{ka : a\epsilon A\} = \begin{cases} [ka^l, ka^u]; \text{if } k \geq 0, \\ [ka^u, ka^l]; \text{if } k < 0, \end{cases}$ where k is a real number.

5. $AB = [\min(S), \max(S)]$, where $S = \{a^l b^l, a^u b^u, a^l b^u, a^u b^l\}$.

The function $F: R^n \to I$ defined on the Euclidean space R^n is called an interval-valued function, i.e., $F(x)$ $F(x) = F(x_1,...,x_n)$ is a closed interval in R for each $x \in R^n$ the interval-valued function F can also be written as $F(x) = [F_l(x), F_u(x)]$, where $F_l(x)$ and $F_u(x)$ are real-valued functions defined on R^n and satisfy $F_l(x) \leq F_u(x)$ for every $x \in R^n$.

Let $C = [c^l, c^u]$ and $D = [d^l, d^u]$ be two closed intervals in R. Let us recall that $C \prec D$ if and only if $c^l < d^l, c^u \leq d^u$ or $c^l \leq d^l, c^u < d^u$ or $c^l < d^l, c^u < d^u$.

($\prec$ is a partial ordering not a total ordering on I).

3. Problem Formulation

The interval-valued fractional continuous static games **(IV-FCSG)$_1$** problem can be written as:

(IV-FCSG)1:

$$Min\, G_i(x,u) = \frac{[G_{il}(x,u)\ , G_{iu}(x,u)]}{[G'_{il}(x,u)\ , G'_{iu}(x,u)]}\ ,\ i = 1,2,...,r, \tag{1}$$

subject to

$$g_j(x,u) = [g_{jl}(x,u), g_{ju}(x,u)] = [0,0],\ j = 1,2,...,n, \tag{2}$$

$$h_k(x,u) = [h_{kl}(x,u), h_{ku}(x,u)] \geq [0,0], k = 1,2,..., q, \tag{3}$$

In **(IV-FCSG)$_1$** problem each player $i = 1,2,...,r$ selects his control vector $u^i \in R^{s_i}$, where $x \in R^n$ is the state control and $u = (u^1, u^2, ..., u^r) \in R^s$, $s = s_1 + s_2 + ... + s_r$ is the composite control. Also $x = \zeta(u)$ is the solution to (2) given u. The functions $G_i(x,u)$ $g_j(x,u)$ and $h_k(x,u)$ are assumed to be in class C^1, with

$$\left| \frac{\partial g(x,u)}{\partial x} \right| \neq 0, \tag{4}$$

in a ball about a solution point (x,u).

The function $G_i(x,u)$ is called an interval-valued function if the coefficients are taken as closed intervals; they will be categorized as interval-valued optimization problems. The functions,

$G_{il}(x,u), G_{iu}(x,u), G'_{il}(x,u), G'_{iu}(x,u), g_{jl}(x,u), g_{ju}(x,u)$ *and* $h_{kl}(x,u), h_{ku}(x,u)$ satisfy $G_{il}(x,u) \le G_{iu}(x,u)$, $G'_{il}(x,u) \le G'_{iu}(x,u)$, $g_{jl}(x,u) \le g_{ju}(x,u)$ *and* $h_{kl}(x,u) \le h_{ku}(x,u)$ for every $(x,u) \in R^n \times R^s$ .and $[G'_{il}(x,u), G'_{iu}(x,u)] > 0$.

In this game, cooperation among all of the players is possible. It is assumed that each player helps the others up to the point of disadvantage to himself. This is the Pareto- minimal (cooperative) solution concept.

The auxiliary interval-valued fractional continuous static games of **(IV-FCSG)₁** problem can be written as:

(IV-FCSG)₂:

$$Min\, G_i(x,u) = \frac{[G_{il}(x,u)\ , G_{iu}(x,u)]}{[G'_{il}(x,u)\ , G'_{iu}(x,u)]}\ , \ i = 1,2,\ldots,r, \tag{5}$$

subject to

$$g_{jl}\ (x,u) = 0 \quad , \ j = 1,2,\ldots,n, \tag{6}$$

$$g_{ju}\ (x,u) = 0, \quad j = 1,2,\ldots,n, \tag{7}$$

$$h_{kl}\ (x,u) \ge 0 \quad , \ k = 1,2,\ldots,q, \tag{8}$$

$$h_{ku}\ (x,u) \ge 0 \quad , \ k = 1,2,\ldots,q. \tag{9}$$

We can transform **(IV-FCSG)₂** problem into the interval-valued non-fractional continuous static game **(IV-NFCSG)** by using Dinkelbach method [10] as the following parametric problem:

(IV-NFCSG):

$$Min\ G'_i(x,u) = [G_{il}(x,u)\ , G_{iu}(x,u)] - \delta_i [G'_{il}(x,u)\ , G'_{iu}(x,u)],\ i = 1,2,\ldots,r, \tag{10}$$

subject to

$$g_{jl}\ (x,u) = 0, \ j = 1,2,\ldots,n, \tag{11}$$

$$g_{ju}\ (x,u) = 0, \ \ j = 1,2,\ldots,n, \tag{12}$$

$$h_{kl}\ (x,u) \ge 0, \ k = 1,2,\ldots,q, \tag{13}$$

$$h_{ku}\ (x,u) \ge 0, \ k = 1,2,\ldots,q. \tag{14}$$

where: $\delta_i = G_i(x^*,u^*) \in R$, i = 1,2,...,r, and (x^*,u^*) is a feasible solution of IV-NFCSG problem.The objective function (10) can be written as:

$$Min\ G_i'(x,u) = \left[G_{il}(x,u) - \delta_i G_{il}'(x,u), G_{iu}(x,u) - \delta_i G_{iu}'(x,u)\right],\ i = 1,2,...,r$$

which can be put in the following form:

$$Min\ G_i'(x,u) = \left[G_i^{\,l}(x,u)\ , G_i^{\,u}(x,u)\ \right],\ i = 1,2,...,r$$

Where $G_i^{\,l}(x,u) = G_{il}(x,u) - \delta_i G_{il}'(x,u)$, $G_i^{\,u}(x,u) = G_{iu}(x,u) - \delta_i G_{iu}'(x,u)$.

So **IV-NFCSG** problem (10)-(14) take the form:

(IV-NFCSG)₁:

$$Min\ G_i'(x,u) = \left[G_i^{\,l}(x,u)\ , G_i^{\,u}(x,u)\ \right],\ i = 1,2,...,r, \tag{15}$$

subject to

$$g_{jl}\ (x,u) = 0 \quad ,\ j = 1,2,...,n, \tag{16}$$

$$g_{ju}\ (x,u) = 0, \quad j = 1,2,...,n, \tag{17}$$

$$h_{kl}\ (x,u) \geq 0 \quad ,\ k = 1,2,...,q, \tag{18}$$

$$h_{ku}\ (x,u) \geq 0 \quad ,\ k = 1,2,...,q. \tag{19}$$

Definition1.

Let (x^*,u^*) be a feasible solution of **(IV-NFCSG)₁** problem, we say that (x^*,u^*) is a non-dominated solution of **(IV-NFCSG)₁** problem if there exist no feasible solution $(\hat{x},\hat{u})$ such that $G_i'(\hat{x},\hat{u}) \prec G_i'(x^*,u^*)$,

i =1, 2....r.

Now we consider the following real-valued optimization problem:

(RV-OP):

$$Min\ G_i''(x,u) = G_i^{\,l}(x,u) + G_i^{\,u}(x,u)\ ,\ i = 1,2,...,r, \tag{20}$$

subject to

$$g_{jl}\ (x,u) = 0 \quad ,\ j = 1,2,...,n, \tag{21}$$

$$g_{ju}\ (x,u) = 0, \quad j = 1,2,...,n, \tag{22}$$

$$h_{kl}\ (x,u) \geq 0 \quad , \ k = 1,2,\ldots,q, \tag{23}$$

$$h_{ku}\ (x,u) \geq 0 \quad , \ k = 1,2,\ldots,q. \tag{24}$$

Then we have the following Proposition.

Proposition1.

If (x^*, u^*) is a Pareto minimal solution of **RV-OP** problem, then (x^*, u^*) is a non- dominated solution of **(IV –NFCSG)$_1$** problem.

Proof

We see that problems **(IV-NFCSG)$_1$** and RV-OP problem have the identical feasible sets. Suppose that (x^*, u^*) is not a non-dominated solution.Then there exist a feasible solution $(\hat{x},\hat{u})$ such that $G_i'(\hat{x},\hat{u}) \prec G_i'(x^*, u^*)$, it means that

$$\begin{cases} G_i^{\,l}(\hat{x},\hat{u}) < G_i^{\,l}(x,^* u^*) \,, G_i^{\,u}(\hat{x},\hat{u}) \leq G_i^{\,u}(x,^* u^*) \quad or \end{cases}$$
$$\begin{cases} G_i^{\,l}(\hat{x},\hat{u}) \leq G_i^{\,l}(x,^* u^*) \,, G_i^{\,u}(\hat{x},\hat{u}) < G_i^{\,u}(x,^* u^*) \quad or \end{cases}$$
$$\begin{cases} G_i^{\,l}(\hat{x},\hat{u}) < G_i^{\,l}(x,^* u^*) \,, G_i^{\,u}(\hat{x},\hat{u}) < G_i^{\,u}(x,^* u^*) \,. \end{cases}$$

It also shows that $G_i''(\hat{x},\hat{u}) < G_{i_i}''(x^*, u^*)$, *i=1,2,….r*, which contradicts the fact that (x^*, u^*) is a Pareto minimal solution of **RV-OP** problem.

Now the real-valued non-fractional continuous static game **(RV-NFCSG)** problem takes the form:

RV-NFCSG:

$$Min\ G_i''\ (x,u) = G_{il}(x,u) - \delta_i\, G_{il}'(x,u) + G_{iu}(x,u) - \delta_i G_{iu}'(x,u)\ \ , \ i = 1,2,\ldots,r, \tag{25}$$

subject to

$$g_{jl}\ (x,u) = 0 \quad , \ j = 1,2,\ldots,n, \tag{26}$$

$$g_{ju}\ (x,u) = 0, \ \ j = 1,2,\ldots,n, \tag{27}$$

$$h_{kl}\ (x,u) \geq 0 \quad , \ k = 1,2,\ldots,q, \tag{28}$$

$$h_{ku}\ (x,u) \geq 0 \quad , \ k = 1,2,\ldots,q. \tag{29}$$

Therefore, the Kuhn-Tuker necessary optimality condition for determining Pareto minimal solution corresponding to **(RV-NFCSG)** problem will have the following form:

$$\frac{\partial L\,(x,u,\eta,\lambda,\lambda',\mu,\mu')}{\partial x}=0,$$

$$\frac{\partial L\,(x,u,\eta,\lambda,\lambda',\mu,\mu')}{\partial u}=0,$$

$$g_{jl}(x,u)=0 \quad , \; j=1,\dots,n,$$

$$g_{ju}(x,u)=0 \quad , \; j=1,\dots,n,$$

$$h_{kl}(x,u)\geq 0 \quad , \; k=1,\dots,q,$$

$$h_{ku}(x,u)\geq 0 \quad , \; k=1,\dots,q,$$

$$\mu\, h_{kl}(x,u)=0, k=1,\dots,q,$$

$$\mu'\, h_{ku}(x,u)=0, k=1,\dots,q,$$

$$\eta_i\geq 0 \quad ,\sum \; \eta_i=1.$$

where L is the Lagrangian function and the vectors $\lambda\in R^n$,$\lambda'\in R^n$, $\mu\in R^q$, $\mu'\in R^q$ are the lagrange multipliars, $\eta\in R^r$,

$$L\,[x,u,\eta,\lambda,\lambda',\mu,\mu']=\sum_{i=1}^{r}\eta_i^T\left(G_i''(\delta_i)\right)-\sum_{j=1}^{n}\lambda_j^T\, g_{jl}(x,u)-\sum_{j=1}^{n}\lambda_j'^T\, g_{ju}(x,u)-\sum_{k=1}^{q}\mu_k^T\, h_{kl}(x,u)$$
$$-\sum_{k=1}^{q}\mu_k'^T h_{ku}(x,u)$$

(x^*,u^*) is Pareto minimal solution for **(RV-NFCSG)** and $x^*=\zeta(u^*)$ is the solution to problems (26) and (27).

Now, we introduce an algorithm for solving **(IV-FCSG)$_1$** problem.

4. Algorithm

Step 1: Convert **(IV-FCSG)$_1$** problem to **RV- NFCSG** problems (25)-(29).

Step2: Let (x_1,u_1) be a feasible solution of RV-NFCSG problem and $\delta_{i1}=G_i(x_1,u_1)$,$i=1,\dots,r$.

Let $M=1$ (M number of iteration) and go to step (3).

Step 3: Use the Pareto- minimal solution concept to solve the following sub-problem:

(SUB)$_M$:

$$Min\, G_i''\;(\delta_{iM})=(G_{il}(x,u)+G_{iu}(x,u))-\delta_{iM}\,(G_{il}'(x,u)+G_{iu}'(x,u))$$

subject to (26) – (29).

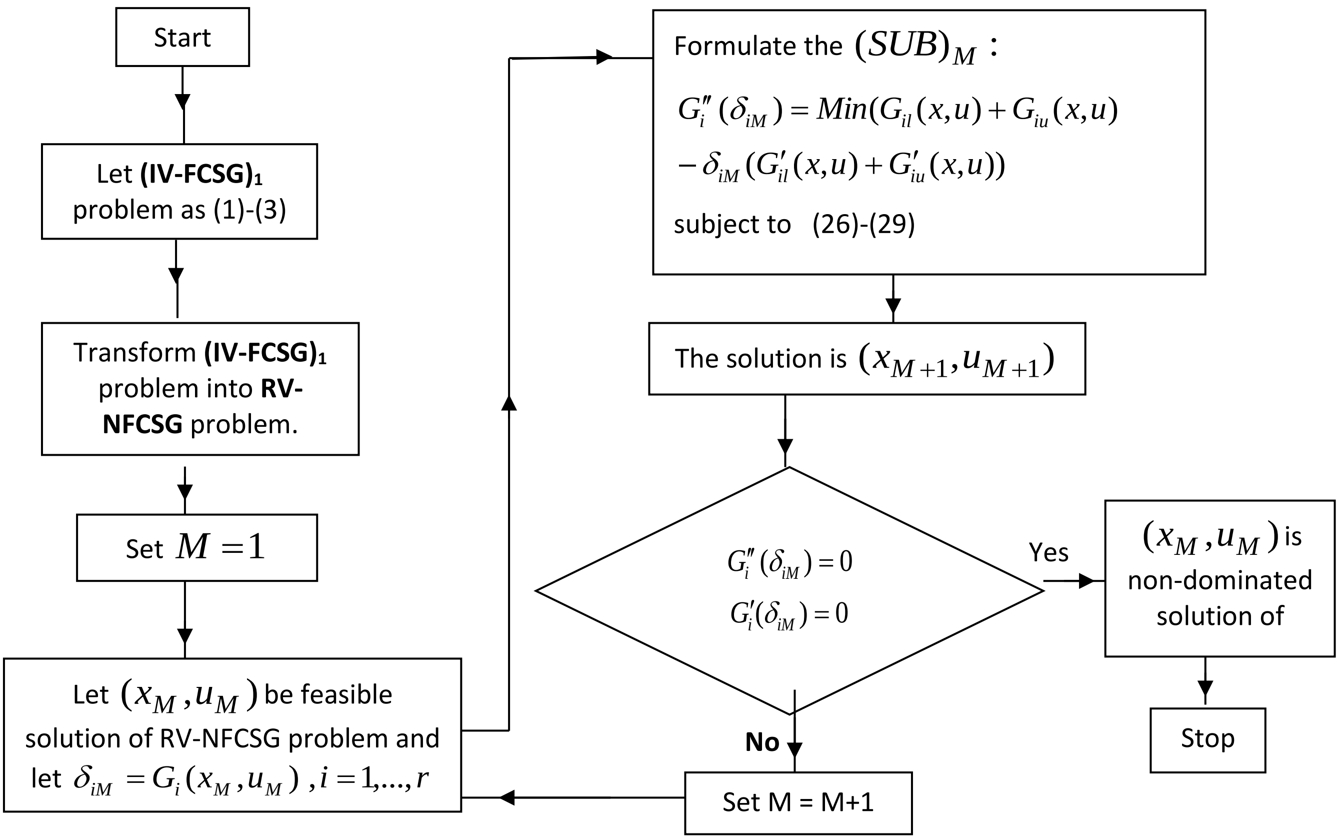
Let a new solution point is (x_{M+1},u_{M+1}).

Step 4: If $G_i''(\delta_{iM}) = 0, \forall i = 1,\ldots,r,$ then (x_M, u_M) is a non-dominated solution of **(IV-FCSG)₁** problem and go to step5. Otherwise set, $\delta_{j(M+1)} = G_j(x_{M+1}, u_{M+1}), j \subseteq i = \{1,2,\ldots,r\}, M = M+1$ and go to step (3).

Step 5: Stop.

In the next section, a flowchart is constructed to explain the algorithm's steps as follows:

5. Flowchart for solving (IFCSG) problem



Figer1.

To demonstrate the solution method of **(IV-FCSG)₁** problem, let us consider the following example.

6. Numerical Example

Consider the following interval fractional continuous static games **IV-FCSG** problem between two-players, where the player (1) selects a control $u \in R^1$ to minimize $G_1(x, u)$, and player (2) selects control $v \in R^1$ to minimize $G_2(x, \upsilon)$ as:

$$Min\ G_1 = \frac{[-1,2]u^2 - [-1,2]\ x}{-[-2,4]\upsilon^2},$$

$$Min\ G_2 = \frac{-[1,2]u^2 + [-1,2]\ x}{[-2,4]\upsilon^2},$$

subject to

$$g = [-1,2]\,x - [0,1]\,u - [-3,4]\ v = [-6,4]$$
$$v \in \left]0.2\right]$$

The above problem can be transform into **IV-NFCSG** as the form:

$$Min\ G_1' = [-u^2 - 2x - 2\delta_{11}v^2,\ 2u^2 + x + 4\delta_{11}\upsilon^2]$$
$$Min\ G_2' = [-2u^2 - x + 2\delta_{21}v^2,\ u^2 + 2x - 4\delta_{21}\upsilon^2]$$

subject to

$$g_1 = -x\ -u - 4v + 6 = 0 \quad ,$$
$$g_2 = 2x\ +3v - 4 = 0.$$
$$0 < v \le 2.$$

Convert **IV-NFCSG** problem to RV-NFCSG Problem as:

$$Min\ G_1''(x_1,u) = u^2 - x + 2\upsilon^2\delta_{11},$$
$$Min\ G_2''(x_2,v) = -u^2 + x - 2\upsilon^2\delta_{21},$$

subject to

$$g_1 = -x\ -u - 4v + 6 = 0 \quad ,$$
$$g_2 = 2x\ +3v - 4 = 0.$$
$$0 < v \le 2.$$

Let $X_1 = (x = 0.5\,,\ u\ = 1.5, v\ = 1)$ be a feasible solution of **RV-NFCS**.

Let $\delta_{11} = G_1(X_1) = -1.625$, $\delta_{21} = G_2(X_1) = -1.625$ and $\eta_1 = 0.25, \eta_2 = 0.75$

For the Pareto minimal solution defines:

$$L = \eta_1(u^2 - x - 3.25\upsilon^2) + \eta_2(-u^2 + x + 3.25\upsilon^2) - \lambda_1(-x - u - 4v + 6) - \lambda_2(2x + 3v - 4.) - \mu_1\upsilon - \mu_2(2 - \upsilon)\ ,$$

Then:

$$\frac{\partial L}{\partial u} = 2\eta_1 u - 2\eta_2 u + \lambda_1 = 0,$$
$$\frac{\partial L}{\partial \upsilon} = -6.5\eta_1\upsilon + 6.5\eta_2\upsilon + 4\lambda_1 - 3\lambda_2 - \mu_1 + \mu_2 = 0,$$
$$\frac{\partial L}{\partial x} = -\eta_1 + \eta_2 + \lambda_1 - 2\lambda_2 = 0$$

$-x\ -u-4v+6=0$,

$2x\ +3v-4=0.$

$v\le 2,\quad \mu_1(2-\upsilon)=0$

$0<v\,,\quad \mu_2\upsilon=0$

$\mu_1\ge 0,\mu_2\ge 0,\eta_1+\eta_2=1,\eta_i\in[0,1],i=1\,,2$

Obtain a new solution $X_2=(x=0.2,\quad u=1,\ v=1.2)$.

So, $\delta_{12}=-0.486$, $\delta_{22}=-0.486.$

By computing $G_1''(X_2)=-0.599\ne 0\,,G_2''\,(X_2)=0.599\ne 0.$

Therefore termination condition is not satisfied with this solution.

Applying step 3 from the algorithm another time at, $\delta_{12}=-0.486$, $\delta_{22}=-0.486.$

We obtain a new solution, $X_3=(-0.1,0.5,1.4)$ as illustrate in the table1.

Iteration M	$X_M=(x,u,v)$	$(\delta_{1M},\quad \delta_{2M})$	$(G_1''(X_M),G_2''(X_M))$
1	$X_1=(0.5,1.5.1)$	(-1.625,-1.625)	(-1.5,1.5)
2	$X_2=(0.2,1,1.2)$	(-0.486,-0.486)	(-0.599,0.599)
3	$X_3=(-0.1,0.5,1.4)$	(-0.05,-0.05)	(0.154,-0.154)
4	$X_4=(0.005,0.675,1.33)$	(-0.137,-0.137)	(0.085,-0.085)
5	$X_5=(0.005,0.675,1.33)$	(-0.109,-0.109)	(0.029,0.029)
6	$X_6=(-5.5\times10^{-3},0.6575,1.337)$	(-0.120,-0.120)	$(8.78969\times10^{-3}$,$-8.78969\times10^{-3})$
7	$X_7=(5\times10^{-4},0.667,1.333)$	(-0.125,-0.125)	$(1.6675\times10^{-4}$,$-1.6675\times10^{-4})\cong(0,0)$

Table1.

At iteration 7, we get $G_1''(X_7)=0, G_2''(X_7)=0$, stop since the termination condition is satisfied with this solution and the non-dominated solution of **IV-FCSG** problem is $X_7=(x=5\times10^{-4},u=0.667,v\ =1.333)$.

7. Conclusions

In this paper, a parametric approach is used to solve fractional continuous static games with interval-valued in the objective function and in the left hand said and right hand said of constraints. The Dinkelbach method is used to transform fractional continuous static games into non- fractional continuous static games. We use the Pareto-minimal solution concept to solve this type of game.

Moreover, an algorithm and flowchart is presented to explain this method. Finally, a numerical example to illustrate the steps of an algorithm. There are many open points for future research, such as:

- It is required to design computer programs for the developed algorithms.

- It is required to continue research in the field of stability analysis of interval continuous static games.

- It is required to continue research in interval-valued stochastic continuous static games.

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