Existence and Uniqueness of Abstract Stochastic fractional-Order Differential Equations

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Abstract

In this paper, the existence and uniqueness about the solution for a class of abstract stochastic fractional-order differential equations

$$^{\mathbf{C}}D_t^{\alpha}\mathbf{u}(t) = A\mathbf{u}(t) + f(t, \mathbf{u}(t)) + \sigma(t, \mathbf{u}(t))\dot{W}(t), \quad 0 \leq t \leq T.$$

$$\mathbf{u}^{(k)}(0) = \mathbf{u}_0^{(k)}, \qquad k = 0, 1, 2, \cdots, m-1.$$

where $\alpha \in (0, 1]$ and f, σ are given functions, are investigated, where the fractional derivative is described in Caputo sense. The fractional calculus, stochastic analysis techniques and the standard $Picard's$ iteration method are used to obtain the required results.

Keywords: Stochastic Differential Equations, Abstract Differential Equations, Fractional-Order Differential Equations, $Picard's$ Iteration Method.

Mathematics Subject Classification: 60H20, 60H10, 60H15, 34K30, 26A33, 43G20, 47G10, 77D09.

1 Introduction

Fractional calculus and fractional-order differential equations have been widely applied in many fields of science and engineering, such as physics ([1]-[2]), chemical ([23]-[24]), etc. For example the nonlinear oscillation of earthquake can be modeled with fractional derivatives [25] and the fluid dynamic traffic model with fractional derivatives [26] can eliminate the deficiency arising from the assumption of continuum traffic flow and others see ([6],[7] and [8]). Actually, the concepts of fractional derivatives are not only generalization of the ordinary derivatives, but also it has been found that they can efficiently and properly describe the behavior of many physical systems (real-life phenomena) more accurately than integer order derivatives.

In recent years, stochastic differential equations have become more and more important and interesting to researchers due to their successful and potential applications in various fields ([9],[10],[11],[12],[13],[14],[27] and [28]), and the basic theories and results of stochastic differential equations can be found in

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[27]. Our main result is studying the existence and uniqueness of solution to equation

$$ {}^{\mathbf{C}}D_t^{\alpha}\mathbf{u}(t) = A\mathbf{u}(t) + f(t, \mathbf{u}(t)) + \sigma(t, \mathbf{u}(t))\dot{W}(t), \quad 0 \leq t \leq T. $$

$$ \mathbf{u}^{(k)}(0) = \mathbf{u}_0^{(k)}, \qquad k = 0, 1, 2, \cdots, m-1, \quad \alpha \in (0, 1] \tag{1} $$

where A is the generator of a strongly continuous semigroup $\{\mathcal{T}(t); t \geq 0\}$ on a Hilbert space $\mathcal{H}$.

2 Materials and Methods

In this section, we give some basic definitions, notations and lemmas which will be used throughout the paper, in order to establish our main results.

Remark 1. *Let $(\Omega, \Im, \mathbb{P})$ be a complete probability space, for a separable Hilbert space $\mathcal{H}$ with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. Then $\mathcal{L}_2(\Omega, \mathcal{H})$ is Hilbert space of $\mathcal{H}$-valued random variables with the inner product $\mathbb{E}(\cdot,\cdot)$ and the norm $(\mathbb{E}\|\cdot\|^2)^{1/2}$ in which $\mathbb{E}$ denotes the expectation.*

Remark 2. *For $y \in \mathcal{L}_2(\Omega, \mathcal{H})$, there holds the following Itô isometry property:*

$$ \mathbb{E} \parallel \int_0^t y(s) dW(s) \parallel^2 = \int_0^t \mathbb{E} \parallel y(s) \parallel^2 ds. \tag{2} $$

where $\{W(t)\}_{t\geq 0}$ is the Wiener (Brownian motion) process

Definition 1. *The Reimann-Liouville fractional derivative of f is defined as*

$$ {}^{\boldsymbol{R}}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\alpha+1-n}} ds $$

where $t > 0$, $n-1 < \alpha < n$, $\Gamma(\cdot)$ stands for the gamma function and $n = [\alpha]+1$ with $[\alpha]$ denotes the integer part of α (see e.g., [29]).

The Reimann-Liouville derivative has certain disadvantages when trying to model real-world phenomena with fractional differential equations. Therefore, we shall introduce a modified fractional differential operator D_*^{α} proposed by M. Caputo in his work on the theory of viscoelasticity.

Definition 2. *The Caputo-type derivative of order α for a function f can be written as*

$$ {}^{\boldsymbol{C}}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}} ds $$

where $t > 0$, $n-1 < \alpha < n$. (see e.g., [29]).

Remark 3. *1. The relationship between the Riemann-Liouville derivative and the Caputo-type derivative can be written as*

$$ {}^{\boldsymbol{C}}D_t^{\alpha} f(t) = {}^{\boldsymbol{R}}D_t^{\alpha} f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0) $$

2. The Caputo-type derivative of a constant is equal to zero.

In this paper, we consider the Caputo-type fractional derivative of order α for a vector-valued function $(\mathbf{u}(t))$, and the initial value problem (IVP) of abstract stochastic fractional-order differential equation (1), where $f(t, \mathbf{u}(t))$ and $\sigma(t, \mathbf{u}(t)) : [0, T] \times \mathcal{R}^d \to \mathcal{R}^d$ and the dimension $d \geq 1$. The term $\dot{W}(t) = \frac{dW}{dt}$ describes a state dependent random noise, $\{W(t)\}_{t\geq 0}$ is a standard scalar Brownian motion or Wiener process defined on a given filtered probability space $(\Omega, \Im, \Im_t, \mathbb{P})$ with a normal filtration $\{\Im_t\}_{t\geq 0}$, which is an increasing and continuous family of σ-algebras of $\Im$, contains the $\mathbb{P}$-null sets, and $W(t)$ is $\Im_t$-measurable for all $t \geq 0$.

Let us recall the definition of the fractional integral operator of order α ([29]) as follows:

$$\mathcal{I}^\alpha g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s) ds, \qquad t > 0. \tag{3}$$

Applying the integral operator (3) to the both sides of initial value problem (1), we can obtain the Volterra integral equation (see e.g., [29])

$$\begin{aligned}\mathbf{u}(t) = \sum_{k=0}^{[\alpha]-1} \frac{t^k}{k!}\mathbf{u}^{(k)}(0) \quad &+ \quad \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} A\mathbf{u}(s) ds \\ &+ \quad \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, \mathbf{u}(s)) ds \\ &+ \quad \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sigma(s, \mathbf{u}(s)) dW(s). \end{aligned} \tag{4}$$

where $n-1 < \alpha < n$ and $t \geq 0$.

Lemma 1. *Every solution of the Volterra integral equation (4) is also a solution of the original initial value problem (1), and vice versa.*

In particular, when $0 < \alpha \leq 1$, the Volterra integral equation (4) can be written as

$$\begin{aligned}\mathbf{u}(t) = \mathbf{u}(0) \quad &+ \quad \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} A\mathbf{u}(s) ds \\ &+ \quad \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, \mathbf{u}(s)) ds \\ &+ \quad \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sigma(s, \mathbf{u}(s)) dW(s). \end{aligned} \tag{5}$$

Lemma 2. *The initial value problem (1) is equivalent to the integral equation*

$$\begin{aligned}\mathbf{u}(t) = \mathbf{u}(0) \quad &+ \quad \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} A\mathbf{u}(s) ds \\ &+ \quad \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, \mathbf{u}(s)) ds \\ &+ \quad \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sigma(s, \mathbf{u}(s)) dW(s), \qquad \alpha \in (0, 1] \end{aligned}$$

In other words, every solution of the integral equation (5) is also a solution of our original initial value problem (1) and vice versa.

Proof. for proof see e.g., [29]. □

Throughout the paper the following non-Lipschitz conditions are assumed and imposed:

(A1) Let $\mathcal{T}(\cdot)$ be a $\mathcal{C}_0$-semigroup generated by the unbounded operator A, let $M = \max_{t\in[0,T]} \parallel \mathcal{T}(t) \parallel_{\mathcal{H}}$.

(A2) The functions f and σ are measurable and continuous in $\mathcal{H}$ for each fixed $t \in [0,T]$ and there exists a bounded function $\mathbb{L} : [0,T] \times [0,\infty) \to [0,\infty]$, $(t,u) \mapsto \mathbb{L}(t,u)$ such that

$$\mathbb{E}(\parallel f(t,x) \parallel^2) + \mathbb{E}(\parallel \sigma(t,x) \parallel^2) \leq \mathbb{L}(t, \mathbb{E}(\parallel x \parallel^2)), \tag{6}$$

for all $t \in \mathcal{R}$ and $x \in \mathcal{L}_2(\Omega, \mathcal{H})$.

(A3) There exists a bounded function $\mathcal{K} : [0,T] \times [0,\infty) \to [0,\infty)$ such that

$$\mathbb{E}(\parallel f(t,x)-f(t,y) \parallel^2)+\mathbb{E}(\parallel \sigma(t,x)-\sigma(t,y) \parallel^2) \leq \mathcal{K}(t, \mathbb{E}(\parallel x-y \parallel^2)), \tag{7}$$

for all $t \in \mathcal{R}$ and $x, y \in \mathcal{L}_2(\Omega, \mathcal{H})$.

Lemma 3. *([30]) If the function $\mathbb{L}(t,u)$ is locally integrable in t for each fixed $u \in [0,\infty)$ and is continuous non-decreasing in u for each fixed $t \in [0,T]$, for all $\lambda > 0$, $\quad u_0 \geq 0$, then the integral equation*

$$u(t) = u_0 + \lambda \int_0^t \mathbb{L}(s, u(s))ds,$$

has a global solution on $[0,T]$.

Lemma 4. *([30]) The function $\mathcal{K}(t,u)$ is locally integrable in t for each fixed $u \in [0,\infty)$ and is continuous non-decreasing in u for each fixed $t \in [0,T]$, for $\mathcal{K}(t,0) = 0$ and $\gamma > 0$, if a non-negative continuous function $\phi(t)$ satisfies*

$$\begin{aligned} \phi(t) &\leq \gamma \int_0^t \mathcal{K}(s, u(s))ds, \quad t \in \mathcal{R} \\ \phi(0) &= 0, \end{aligned}$$

then $\phi(t) = 0$ for all $t \in [0,T]$.

In order to consider the existence and uniqueness of the solution of equation (5), we attempt to use the $Picard's$ iteration method. The sequence of stochastic process $\{\mathbf{u}_n\}_{n\geq 0}$ is constructed as follows:

$$\begin{aligned} \mathbf{u}_0(t) &= \mathbf{u}_0, \\ \mathbf{u}_{n+1}(t) &= \mathcal{T}(t)\mathbf{u}_0 + G_1(\mathbf{u}_n)(t) + G_2(\mathbf{u}_n)(t), \quad n \geq 1 \end{aligned}$$

in which

$$\begin{aligned} G_1(\mathbf{u}_n)(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \mathcal{T}(t-s) f(s, \mathbf{u}_n(s))ds \\ G_2(\mathbf{u}_n)(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \mathcal{T}(t-s) \sigma(s, \mathbf{u}_n(s))dW(s) \end{aligned}$$

Lemma 5. *The sequence of stochastic processes $\{\mathbf{u}_n\}_{n\geq 0}$ is bounded in $\mathcal{L}_2(\Omega,\mathcal{H})$.*

Proof. from the inequality

$$(a+b+c)^n \leq 3^{n-1}(a^n+b^n+c^n), \quad n\geq 1.$$

we have

$$\mathbb{E}\parallel \mathbf{u}_{n+1}(t)\parallel^2 \leq 3\mathbb{E}\parallel \mathbf{u}_0\parallel^2 + 3\mathbb{E}\parallel G_1(\mathbf{u}_n)(t)\parallel^2 + 3\mathbb{E}\parallel G_2(\mathbf{u}_n)(t)\parallel^2 . \tag{8}$$

Using the $H\ddot{o}lder's$ inequality, the assumption $(A2)$ and $\alpha > 1/2$, we can obtain

$$\begin{aligned}
\mathbb{E}\parallel G_1(\mathbf{u}_n)(t)\parallel^2 &= \frac{1}{\Gamma^2(\alpha)}\mathbb{E}\parallel \int_0^t (t-s)^{\alpha-1}\mathcal{T}(t-s)f(s,\mathbf{u}_n(s))ds\parallel^2 \\
&\leq \frac{M^2}{\Gamma^2(\alpha)}\cdot\frac{t^{2\alpha-1}}{2\alpha-1}\int_0^t \mathbb{E}\parallel f(s,\mathbf{u}_n(s))\parallel^2 ds \\
&\leq k_1\int_0^t \mathbb{L}(s,\parallel \mathbf{u}_n(s)\parallel^2_{\mathcal{L}_2(\Omega,\mathcal{H})})ds,
\end{aligned}$$

where $k_1 = \frac{M^2T^{2\alpha-1}}{\Gamma^2(\alpha)(2\alpha-1)}$.

Applying the Itô isometry property (2), the $H\ddot{o}lder's$ inequality and the assumptions $(A2)$ and $\alpha > 1/2$, we have

$$\mathbb{E}\parallel G_2(\mathbf{u}_n)(t)\parallel^2 \leq k_1\int_0^t \mathbb{L}(s,\parallel \mathbf{u}_n(s)\parallel^2_{\mathcal{L}_2(\Omega,\mathcal{H})})ds$$

Therefore, using the above relations into the inequality (8), we have

$$\parallel \mathbf{u}_{n+1}(t)\parallel^2_{\mathcal{L}_2(\Omega,\mathcal{H})} \leq c_1 + c_2\int_0^t \mathbb{L}(s,\parallel \mathbf{u}_n(s)\parallel^2_{\mathcal{L}_2(\Omega,\mathcal{H})}), \tag{9}$$

in which $c_1 = 3\mathbb{E}\parallel \mathbf{u}_0\parallel^2$ and $c_2 = 6k_1$.

Then, we consider the following integral equation:

$$x(t) = c_1 + c_2\int_0^t \mathbb{L}(s,x(s))ds, \tag{10}$$

This equation has a globe solution via the Lemma (3).
And we can use the mathematical induction to prove $\parallel \mathbf{u}_n(t)\parallel^2_{\mathcal{L}_2(\Omega,\mathcal{H})} \leq x(t)$ for all $t\in[0,T]$. Particularly, we have $\sup_{n\geq 0}\parallel \mathbf{u}_n(t)\parallel_{\mathcal{L}_2(\Omega,\mathcal{H})} \leq [x(T)]^{1/2}$ □

Lemma 6. *The sequence of stochastic processes $\{\mathbf{u}_n\}_{n\geq 0}$ is a Cauchy sequence.*

3 Main Result

Theorem 1. *Under the conditions (6) and (7), by using lemma (3) and lemma (4), there exists a unique solution of equation (5).*

Proof. **Existence:** If we denote $\mathbf{u}(t)$ by the limit of the sequence $\{\mathbf{u}_n(t)\}_{n\geq 0}$ and by using lemma (6) then we can see that the right hand side in the second $Picard's$ iteration tends to

$$\begin{aligned}\mathcal{T}(t)\mathbf{u}_0 \quad &+ \quad \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\mathcal{T}(t-s)f(s,\mathbf{u}(s))ds \\ &+ \quad \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\mathcal{T}(t-s)\sigma(s,\mathbf{u}(s))dW(s).\end{aligned}$$

which is just a solution of equation (5).
Uniqueness: Let $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are two $solution's$ of equation (5), using lemma (5), we have

$$\| \mathbf{u}(t) - \mathbf{v}(t) \|^2_{\mathcal{L}_2(\Omega,\mathcal{H})} \leq c_3 \int_0^t \mathcal{K}(s, \| \mathbf{u}(s) - \mathbf{v}(s) \|^2_{\mathcal{L}_2(\Omega,\mathcal{H})})ds.$$

Using lemma (3), we can obtain $\| \mathbf{u}(t) - \mathbf{v}(t) \|^2_{\mathcal{L}_2(\Omega,\mathcal{H})} = 0$ for all $t \in [0,T]$, which implies that $\mathbf{u}(t) \equiv \mathbf{v}(t)$. □

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