Inverse scattering with non-over-determined data

Alexander G. Ramm
Department of Mathematics, Kansas State University,
Manhattan, KS 66506, USA
ramm@math.ksu.edu
http://www.math.ksu.edu/ ∼ ramm

Abstract

The results of the author's theory of the inverse scattering with non-over-determined data are described.

1 Introduction

There is a large literature on inverse scattering, see [1] and references therein. We consider the potential scattering and the obstacle scattering.

The potential scattering problem consists of finding the scattering solution $u(x,\alpha,k)$:

$$[\nabla^2 + k^2 - q(x)]u = 0 \quad in \quad \mathbb{R}^3, \tag{1}$$

$$u = u_0 + v, \quad u_0 = e^{ik\alpha\cdot x} \tag{2}$$

$$v_r - ikv = O(r^{-2}), \quad r \to \infty. \tag{3}$$

Here $r := |x|$, $\alpha \in S^2$, S^2 is the unit sphere in $\mathbb{R}^3$, $q = q(x) \in L^2_{loc}(\mathbb{R}^3)$ is assumed to be real-valued and compactly supported. One has

$$v(x,\alpha,k) = A(\beta,\alpha,k)\frac{e^{ikr}}{r} + O(r^{-2}), \quad r \to \infty,\ \beta = x/r. \tag{4}$$

The $A(\beta,\alpha,k)$ is called the scattering amplitude, $\beta \in S^2$ is the direction of the scattered wave.

The inverse scattering problem consists of finding $q(x)$ from the scattering amplitude A. The function A is a function of five variables. It is easy to prove that this function known for all α, $\beta \in S^2$ and $\forall k > 0$ determines q uniquely. In 1987 the author proved that a compactly supported potential q is uniquely determined by the fixed-energy scattering amplitude. More precisely, the values of $A(\beta,\alpha,k_0)$ for β and α running through fixed open subsets of S^2 and $k = k_0 > 0$ fixed determine a compactly supported q uniquely, see [3], [4], [5], [1]. The author also gave stability estimates for q in terms of the scattering amplitude, see [6], [1] and references therein.

MSC: 35P25, 45Q0581U40.
Key words: inverse scattering with non-over-determined data.

Date of Submission: 2019-01-21
Date of Publication: 2019-01-30

However, the fixed-energy data is a function of four variable, while the $q(x)$ is a function of three variables. The non-over-determined data are the values of the scattering amplitude which form a three-dimensional set. For example, the values $A(-\alpha, \alpha, k)$ for all $\alpha \in S^2$ and all $k > 0$ is such a set. These are the back-scattering data at all energies. For compactly supported potentials the author proved uniqueness of the solution to the inverse scattering problem with the non-over-determined data $A(-\alpha, \alpha, k)$ known for all k in an arbitrary small open subset of $[0, \infty)$ and all α in an arbitrary small open subset of S^2. The author proved that for a compactly supported potential these data determine uniquely the values of $A(-\alpha, \alpha, k)$ for all $k > 0$ and all $\alpha \in S^2$.

The other practically interesting example of non-over-determined data for which the author proved the uniqueness of the solution to the inverse scattering problem are the values of $A(\beta, \alpha_0, k)$ known for all k in an arbitrary small open subset of $[0, \infty)$ and all β in an arbitrary small open subset of S^2, $\alpha = \alpha_0$ being fixed.

These results are first published in [13], [14], [15] and in the monograph [1].

The obstacle scattering problem consists of finding the scattering solution $u(x, \alpha, k)$. Let $D \subset \mathbb{R}^3$ be a bounded domain with a smooth connected boundary S, $D' := \mathbb{R}^3 \setminus D$. Then

$$(\nabla^2 + k^2)u = 0 \quad in \quad D', \quad u|_S = 0, \tag{5}$$

$$u = u_0 + v, \quad u_0 = e^{ik\alpha\cdot x} \tag{6}$$

$$v_r - ikv = O(r^{-2}), \quad r \to \infty. \tag{7}$$

One has

$$v(x, \alpha, k) = A(\beta, \alpha, k)\frac{e^{ikr}}{r} + O(r^{-2}), \quad r \to \infty, \ \beta = x/r. \tag{8}$$

The non-over-determined data are the values of $A(\beta, \alpha, k)$ on a two-dimensional subset of the set $S^2 \times S^2 \times [0, \infty)$. For example, such is the set $\forall \beta \in S^2$, a fixed $\alpha = \alpha_0$ and a fixed $k = k_0 > 0$.

The author proved that these non-over-determined data determine uniquely the surface S and the boundary condition on S.

The boundary condition is assumed of the Dirichle, or Neumann, or impedance type. The impedance boundary condition is

$$u_N = \zeta u \quad on \quad S. \tag{9}$$

Here $\zeta = \zeta(s)$ is the boundary impedance and it is assumed that

$$Im\zeta \leq 0. \tag{10}$$

Assumption (10) guarantees uniqueness of the solution to the obstacle scattering problem, [11].

The uniqueness theorems for inverse obstacle scattering with non-over-determined data is proved by the author in [8], [1], [16].

We now sketch this proof.

Let us assume that two obstacles D_1 and D_2 generate the same scattering amplitude for all $\beta \in S^2$, a fixed α and a fixed $k = k_0 > 0$, and prove that then $D_1 = D_2$ and the boundary conditions are the same. If $D_1 = D_2 := D$ then $u_1 = u_2$ in D', so $u_1 = u_2$ and $U_{1N} = u_{2N}$ on $S := \partial D$. Consequently, the boundary conditions are the same.

Let us prove that $S_1 = S_2$ if $A_1(\beta) = A_2(\beta)$ for all $\beta \in S^2$, where $A_j(\beta) := A_j(\beta, \alpha_0, k_0)$, $j = 1, 2$.

If $A_1(\beta) = A_2(\beta)$ then $u_1(x, \alpha_0, k_0) = u_2(x, \alpha_0, k_0)$ for all $x \in D'_{12} := \mathbb{R}^3 \setminus (D_1 \cup D_2)$.
This follows from Lemma 1.2.15 in [1], p.47. Let $D^{12} := D_1 \cap D_2$, $S_{12} := \partial D_{12}$, $S^{12} := \partial D^{12}$.
One has $u_1 = u_2 := u$ in $\mathbb{R}^3 \setminus D^{12}$.
By Green's formula one gets

$$u = u_0 - \int_{S_1} g(x,s) u_N ds, \quad x \in D'_1 \tag{11}$$

and

$$u = u_0 - \int_{S_2} g(x,s) u_N ds, \quad x \in D'_2. \tag{12}$$

Since u and u_0 are defined in $\mathbb{R}^3 \setminus D^{12}$, the integrals in (11) and (12) are also defined in $\mathbb{R}^3 \setminus D^{12}$. Consequently one obtains

$$\int_{S_1} g(x,s) u_N ds = \int_{S_2} g(x,s) u_N ds \quad x \in D_{12} \setminus D^{12}. \tag{13}$$

By Green's formula one has

$$u(x) = \int_{S_2} g(x,s) u_N ds - \int_{S_1} g(x,s) u_N ds = 0, \quad x \in D_{12} \setminus D^{12}, \tag{14}$$

where formula (13) was used.

Since u is an analytic function of x in $\mathbb{R}^3 \setminus D^{12}$ and vanishes in $D_{12} \setminus D^{12}$ it must vanish everywhere in $\mathbb{R}^3 \setminus D^{12}$.

This is a contradiction since

$$\lim_{|x| \to \infty} |u(x, \alpha_0, k_0)| = 1.$$

This contradiction proves that $D_1 = D_2$, so $S_1 = S_2$. The proof is complete. $\Box$

A study of the inverse scattering problems with non-over-determined data is of principal interest because these are the minimal data from which the unknown scatterer can be uniquely determined.

References

[1] A.G.Ramm, **Scattering by obstacles and potentials**, World Sci. Publ., Singapore, 2017.

[2] A.G.Ramm, A uniqueness theorem in scattering theory. Phys, Rev. Lett., 52, N1, (1984), 13.

[3] A.G.Ramm, Completeness of the products of solutions to PDE and uniqueness theorems in inverse scattering, Inverse problems, 3, (1987), L77-L82

[4] A.G.Ramm, Multidimensional inverse problems and completeness of the products of solutions to PDE, J. Math. Anal. Appl. 134, 1, (1988), 211-253; 139, (1989) 302.

[5] A.G.Ramm, Completeness of the products of solutions of PDE and inverse problems, Inverse Probl.6, (1990), 643-664.

[6] A.G.Ramm, Stability of solutions to inverse scattering problems with fixed-energy data, Milan Journ of Math., 70, (2002), 97-161.

[7] A.G.Ramm, Inverse scattering with under-determined scattering data, Math. Meth. in Natur. Phenomena, (MMNP), 9, N5, (2014), 244-253.

[8] A.G.Ramm, Uniqueness of the solution to inverse obstacle scattering with non-over-determined data, Appl. Math. Lett., 58, (2016), 81-86.

[9] A.G.Ramm, A numerical method for solving 3D inverse scattering problem with non-over-determined data, J. Pure Appl. Math., 1, N1, (2017), 1-3.

open access Journal

[10] A.G.Ramm, On the denseness of the set of scattering amplitudes, International Review of Physics, 11, N4, (2017), 96-98.

[11] A.G.Ramm, **Creating materials with a desired refraction coefficient,** IOP Concise Physics, Morgan and Claypool Publishers, San Rafael, CA, USA, 2017.

[12] A.G.Ramm, Cong Van, A numerical algorithm for solving 3D inverse scattering problem with non-over-determined data, J. Appl. Math. Stat. App., 2, N1, (2018), 11-13.

open access Journal

[13] A.G.Ramm, Uniqueness theorem for inverse scattering problem with non-overdetermined data, J.Phys. A, FTC, 43, (2010), 112001.

[14] A.G.Ramm, Uniqueness of the solution to inverse scattering problem with backscattering data, Eurasian Math. Journ (EMJ), 1, N3, (2010), 97-111.

open access Journal.

[15] A.G.Ramm, Uniqueness of the solution to inverse scattering problem with scattering data at a fixed direction of the incident wave, J. Math. Phys., 52, 123506, (2011).

[16] A.G.Ramm, Inverse obstacle scattering with non-over-determined data, Global Journ. of Math. Anal. (GJMA), 6 (1), (2018), 2–6.