Solutions of Some Difference Equations Systems and Periodicity

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Abstract

In this article, analysis and investigation have been conducted on the periodic nature as well as the type of the solutions of the subsequent schemes of rational difference equations

$$x_{n+1} = \frac{1 \pm z_n}{y_{n-1}},\ y_{n+1} = \frac{1 \pm x_n}{z_{n-1}}, z_{n+1} = \frac{1 \pm y_n}{x_{n-1}},$$

with a nonzero real numbers initial conditions.

Keywords: difference equation, periodic solutions, system of difference equations.
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1 Introduction

Difference equations have normally been shown as discrete analogues as well as numerical solutions of differential and delay differential equations having some important uses in scientific areas such as, ecology, physics, economy, biology, etc. Currently, expanding concern has obviously been conducted on the study of qualitative analysis of rational difference equations and systems of difference equations. Even though, in form, difference equations look like to be elementary, it is quit complicated to be analyzed and understood thoroughly the nature of their solutions. see [1]–[23] and the references cited therein.

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A great number of researchers have examined periodic solutions of a difference equations , and different approaches have been provided for the existence and qualitative properties of the solutions.

$$x_{n+1}=\frac{m}{y_n},\quad y_{n+1}=\frac{py_n}{x_{n-1}y_{n-1}},$$

was studied by Cinar in [5].
Elsayed [12] has got the solutions of the following systems of the difference equations

$$x_{n+1}=\frac{x_{n-1}}{\pm 1+x_{n-1}y_n},\ y_{n+1}=\frac{y_{n-1}}{\mp 1+y_{n-1}x_n}.$$

Liu et al. [24] obtained the solution of particular cases of the following general system of difference equations

$$x_{n+1}=\tfrac{x_{n-1}}{y_nx_{n-1}-1},\ y_{n+1}=\tfrac{y_{n-1}}{x_ny_{n-1}-1},\ z_{n+1}=\tfrac{1}{y_nz_{n-1}}.$$

Özban [25] has investigated the positive solutions of the system of rational difference equations

$$x_{n+1}=\frac{a}{y_{n-3}},\ y_{n+1}=\frac{by_{n-3}}{x_{n-q}y_{n-q}}.$$

In [29] Yalçınkaya investigated the sufficient condition for the global asymptotic stability of the following system of difference equations

$$z_{n+1}=\frac{t_nz_{n-1}+a}{t_n+z_{n-1}},\ t_{n+1}=\frac{z_nt_{n-1}+a}{z_n+t_{n-1}}.$$

Also, Yalçınkaya [30] has obtained the sufficient conditions for the global asymptotic stability of the system of two nonlinear difference equations

$$x_{n+1}=\frac{x_n+y_{n-1}}{x_ny_{n-1}-1},\ y_{n+1}=\frac{y_n+x_{n-1}}{y_nx_{n-1}-1}.$$

Zhang et al.[36] has investigated the positive solutions of the systems

$$x_{n+1}=A+\frac{y_{n-k}}{y_n},\quad y_{n+1}=A+\frac{x_{n-k}}{x_n}.$$

Similar nonlinear systems of rational difference equations were investigated see [24]-[34].

Our aim in this paper is to investigate the periodic nature and get the form of the solutions of the following systems of rational difference equations

$$x_{n+1}=\frac{1\pm z_n}{y_{n-1}},\quad y_{n+1}=\frac{1\pm x_n}{z_{n-1}},\quad z_{n+1}=\frac{1\pm y_n}{x_{n-1}}.$$

with a nonzero real numbers initial conditions.
Definition (Periodicity)

A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $x_{n+p}=x_n$ for all $n\geq -k$.

2 First System: $x_{n+1} = \dfrac{1+z_n}{y_{n-1}}, y_{n+1} = \dfrac{1+x_n}{z_{n-1}}, z_{n+1} = \dfrac{1+y_n}{x_{n-1}}$.

In this part, the solutions of the system of two difference equations have been studied

$$x_{n+1} = \frac{1+z_n}{y_{n-1}}, \quad y_{n+1} = \frac{1+x_n}{z_{n-1}}, \quad z_{n+1} = \frac{1+y_n}{x_{n-1}}, \qquad n = 0, 1, ..., \tag{1}$$

with nonzero real initials conditions $x_{-1},\ x_0,\ y_{-1},\ y_0,\ z_{-1},\ z_0$.

Theorem 1 *Suppose that $\{x_n, y_n, z_n\}$ are solutions of system (1), then the following statements are true:-*

1. $x_{n+5} = y_n,\ y_{n+5} = z_n,\ z_{n+5} = x_n$ for $n \geq -1$.
2.$x_{n+10} = z_n,\ y_{n+10} = x_n,\ z_{n+10} = y_n$ for $n \geq -1$.
3. $\{x_n\}_{n=-1}^{+\infty}$, $\{y_n\}_{n=-1}^{+\infty}$and $\{z_n\}_{n=-1}^{+\infty}$ are periodic with period ten i.e., $x_{n+15} = x_n,\ y_{n+15} = y_n,\ z_{n+15} = z_n$ for $n \geq -1$.
4. We have

$$\begin{aligned}
x_{10n-1} &= x_{-1}, \quad x_{10n} = x_0, \quad x_{10n+1} = \frac{1+z_0}{y_{-1}}, \quad x_{10n+2} = \frac{1+y_0+x_{-1}}{y_0 x_{-1}}, \\
x_{10n+3} &= \frac{1+z_{-1}}{x_0}, \quad x_{10n+4} = y_{-1}, \quad x_{10n+5} = y_0, \quad x_{10n+6} = \frac{1+x_0}{z_{-1}}, \\
x_{10n+7} &= \frac{1+z_0+y_{-1}}{z_0 y_{-1}}, \quad x_{10n+8} = \frac{1+x_{-1}}{y_0}, \quad x_{10n+9} = z_{-1}, \quad x_{10n+10} = z_{0,} \\
x_{10n+11} &= \frac{1+y_0}{x_{-1}}, \quad x_{10n+12} = \frac{1+x_0+z_{-1}}{x_0 z_{-1}}, \quad x_{10n+13} = \frac{1+y_{-1}}{z_0}.
\end{aligned}$$

and

$$\begin{aligned}
y_{10n-1} &= y_{-1}, \quad y_{10n} = y_0, \quad y_{10n+1} = \frac{1+x_0}{z_{-1}}, \quad y_{10n+2} = \frac{1+z_0+y_{-1}}{z_0 y_{-1}}, \\
y_{10n+3} &= \frac{1+x_{-1}}{y_0}, \quad y_{10n+4} = z_{-1}, \quad y_{10n+5} = z_0, \quad y_{10n+6} = \frac{1+y_0}{x_{-1}}, \\
y_{10n+7} &= \frac{1+x_0+z_{-1}}{x_0 z_{-1}}, \quad y_{10n+8} = \frac{1+y_{-1}}{z_0}, \quad y_{10n+9} = x_{-1}, \quad y_{10n+10} = x_{0,} \\
y_{10n+11} &= \frac{1+z_0}{y_{-1}}, \quad y_{10n+12} = \frac{1+y_0+x_{-1}}{y_0 x_{-1}}, \quad y_{10n+13} = \frac{1+z_{-1}}{x_0}.
\end{aligned}$$

as well

$$z_{10n-1} = z_{-1}, \quad z_{10n} = z_0, \quad z_{10n+1} = \frac{1+y_0}{x_{-1}}, \quad z_{10n+2} = \frac{1+x_0+z_{-1}}{x_0 z_{-1}},$$
$$z_{10n+3} = \frac{1+y_{-1}}{z_0}, \quad z_{10n+4} = x_{-1}, \quad z_{10n+5} = x_0, \quad z_{10n+6} = \frac{1+z_0}{y_{-1}},$$
$$z_{10n+7} = \frac{1+y_0+x_{-1}}{y_0 x_{-1}}, \quad z_{10n+8} = \frac{1+z_{-1}}{x_0}, \quad z_{10n+9} = y_{-1}, \quad z_{10n+10} = y_0,$$
$$z_{10n+11} = \frac{1+x_0}{z_{-1}}, \quad z_{10n+12} = \frac{1+z_0+y_{-1}}{z_0 y_{-1}}, \quad z_{10n+13} = \frac{1+x_{-1}}{y_0}.$$

Or equivalently

$$\{x_n\}_{n=-1}^{+\infty} = \left\{ \begin{array}{c} x_{-1}, x_0, \frac{1+z_0}{y_{-1}}, \frac{1+y_0+x_{-1}}{y_0 x_{-1}}, \frac{1+z_{-1}}{x_0}, y_{-1}, y_0, \frac{1+x_0}{z_{-1}}, \frac{1+z_0+y_{-1}}{z_0 y_{-1}}, \\ \frac{1+x_{-1}}{y_0}, z_{-1}, z_0, \frac{1+y_0}{x_{-1}}, \frac{1+x_0+z_{-1}}{x_0 z_{-1}}, \frac{1+y_{-1}}{z_0}, x_{-1}, x_0, \ldots \end{array} \right\},$$
$$\{y_n\}_{n=-1}^{+\infty} = \left\{ \begin{array}{c} y_{-1}, y_{-1}, \frac{1+x_0}{z_{-1}}, \frac{1+z_0+y_{-1}}{z_0 y_{-1}}, \frac{1+x_{-1}}{y_0}, z_{-1}, z_0, \frac{1+y_0}{x_{-1}}, \frac{1+x_0+z_{-1}}{x_0 z_{-1}}, \\ \frac{1+y_{-1}}{z_0}, x_{-1}, x_0, \frac{1+z_0}{y_{-1}}, \frac{1+y_0+x_{-1}}{y_0 x_{-1}}, \frac{1+z_{-1}}{x_0}, y_{-1}, y_0, \ldots \end{array} \right\},$$
$$\{z_n\}_{n=-1}^{+\infty} = \left\{ \begin{array}{c} z_{-1}, z_0, \frac{1+y_0}{x_{-1}}, \frac{1+x_0+z_{-1}}{x_0 z_{-1}}, \frac{1+y_{-1}}{z_0}, x_{-1}, x_0, \frac{1+z_0}{y_{-1}}, \frac{1+y_0+x_{-1}}{y_0 x_{-1}}, \\ \frac{1+z_{-1}}{x_0}, y_{-1}, y_0, \frac{1+x_0}{z_{-1}}, \frac{1+z_0+y_{-1}}{z_0 y_{-1}} \frac{1+x_{-1}}{y_0}, z_{-1}, z_0, \ldots \end{array} \right\}.$$

Proof: 1. From Eq.(1) we see that

$$x_{n+5} = \frac{1+z_{n+4}}{y_{n+3}}, \qquad y_{n+5} = \frac{1+x_{n+4}}{z_{n+3}}, \quad z_{n+5} = \frac{1+y_{n+4}}{x_{n+3}}$$
$$x_{n+5} = \frac{1+\left(\frac{1+y_{n+3}}{x_{n+2}}\right)}{\left(\frac{1+x_{n+2}}{z_{n+1}}\right)}, \qquad y_{n+5} = \frac{1+\left(\frac{1+z_{n+3}}{y_{n+2}}\right)}{\left(\frac{1+y_{n+2}}{x_{n+1}}\right)}, y_{n+5} = \frac{1+\left(\frac{1+x_{n+3}}{z_{n+2}}\right)}{\left(\frac{1+z_{n+2}}{y_{n+1}}\right)}$$

$$x_{n+5} = \frac{1+z_{n+1}}{x_{n+2}}, \qquad y_{n+5} = \frac{1+x_{n+1}}{y_{n+2}}, \quad z_{n+5} = \frac{1+y_{n+1}}{z_{n+2}},$$
$$x_{n+5} = \frac{1+z_{n+1}}{\frac{1+z_{n+1}}{yn}}, \quad y_{n+5} = \frac{1+x_{n+1}}{\frac{1+x_{n+1}}{z_n}}, \quad z_{n+5} = \frac{1+y_{n+1}}{\frac{1+y_{n+1}}{x_n}}.$$

Therefore

$$x_{n+5} = y_n, \quad y_{n+5} = z_n, \quad z_{n+5} = x_n.$$

2. Also, we get

$$x_{n+10} = y_{n+5} = z_n,$$
$$y_{n+10} = z_{n+5} = x_n,$$

$$z_{n+10} = x_{n+5} = y_n.$$

3. Ditto,

$$x_{n+15} = y_{n+10} = z_{n+5} = x_n,$$

$$y_{n+15} = z_{n+10} = x_{n+5} = y_n,$$

$$z_{n+10} = x_{n+10} = y_{n+5} = z_n.$$

3. For $n = 0$ the result holds. Now suppose that $n > 0$ and that our assumption holds for $n - 1$. That is;

$$\begin{aligned}
x_{15n-16} &= x_{-1}, \quad x_{15n-15} = x_0, \quad x_{15n-14} = \frac{1+z_0}{y_{-1}}, \quad x_{15n-13} = \frac{1+y_0+x_{-1}}{y_0 x_{-1}}, \\
x_{15n-12} &= \frac{1+z_{-1}}{x_0}, \quad x_{15n-11} = y_{-1}, \quad x_{15n-10} = y_0, \quad x_{15n-9} = \frac{1+x_0}{z_{-1}}, \\
x_{15n-8} &= \frac{1+z_0+y_{-1}}{z_0 y_{-1}}, \quad x_{15n-7} = \frac{1+x_{-1}}{y_0}, \quad x_{15n-6} = z_{-1}, \quad x_{15n-5} = z_0, \\
x_{15n-4} &= \frac{1+y_0}{x_{-1}}, \quad x_{15n-3} = \frac{1+x_0+z_{-1}}{x_0 z_{-1}}, \quad x_{15n-2} = \frac{1+y_{-1}}{z_0}.
\end{aligned}$$

and

$$\begin{aligned}
y_{15n-16} &= y_{-1}, \quad y_{15n-15} = y_0, \quad y_{15n-14} = \frac{1+x_0}{z_{-1}}, \quad y_{15n-13} = \frac{1+z_0+y_{-1}}{z_0 y_{-1}}, \\
y_{15n-12} &= \frac{1+x_{-1}}{y_0}, \quad y_{15n-11} = z_{-1}, \quad y_{15n-10} = z_0, \quad y_{15n-9} = \frac{1+y_0}{x_{-1}}, \\
y_{15n-8} &= \frac{1+x_0+z_{-1}}{x_0 z_{-1}}, \quad y_{15n-7} = \frac{1+y_{-1}}{z_0}, \quad y_{15n-6} = x_{-1}, \quad y_{15n-5} = x_0, \\
y_{15n-4} &= \frac{1+z_0}{y_{-1}}, \quad y_{15n-3} = \frac{1+y_0+x_{-1}}{y_0 x_{-1}}, \quad y_{15n-2} = \frac{1+z_{-1}}{x_0}.
\end{aligned}$$

and the result of Z

$$\begin{aligned}
z_{15n-16} &= z_{-1}, \quad z_{15n-15} = z_0, \quad z_{15n-14} = \frac{1+y_0}{x_{-1}}, \quad z_{15n-13} = \frac{1+x_0+z_{-1}}{x_0 z_{-1}}, \\
z_{15n-12} &= \frac{1+y_{-1}}{z_0}, \quad z_{15n-11} = x_{-1}, \quad z_{15n-10} = x_0, \quad z_{15n-9} = \frac{1+z_0}{y_{-1}}, \\
z_{15n-8} &= \frac{1+y_0+x_{-1}}{y_0 x_{-1}}, \quad z_{15n-7} = \frac{1+z_{-1}}{x_0}, \quad z_{15n-6} = y_{-1}, \quad z_{15n-5} = y_0, \\
z_{15n-4} &= \frac{1+x_0}{z_{-1}}, \quad z_{15n-3} = \frac{1+z_0+y_{-1}}{z_0 y_{-1}}, \quad z_{15n-2} = \frac{1+x_{-1}}{y_0}.
\end{aligned}$$

Now, it follows from Eq.(1) that

$$
\begin{aligned}
x_{15n-1} &= \frac{1+z_{15n-2}}{y_{15n-3}} = \frac{1+\left(\frac{1+x_{-1}}{y_0}\right)}{\left(\frac{1+y_0+x_{-1}}{y_0x_{-1}}\right)} = \frac{(1+y_0+x_{-1})\,x_{-1}}{1+y_0+x_{-1}} = x_{-1},\\
y_{15n-1} &= \frac{1+x_{15n-2}}{z_{15n-3}} = \frac{1+\left(\frac{1+y_{-1}}{z_0}\right)}{\left(\frac{1+z_0+y_{-1}}{z_0y_{-1}}\right)} = \frac{(1+z_0+y_{-1})\,y_{-1}}{1+z_0+y_{-1}} = y_{-1},\\
z_{15n-1} &= \frac{1+y_{15n-2}}{x_{15n-3}} = \frac{1+\left(\frac{1+z_{-1}}{x_0}\right)}{\left(\frac{1+x_0+z_{-1}}{x_0z_{-1}}\right)} = \frac{(1+x_0+z_{-1})\,z_{-1}}{1+x_0+z_{-1}} = z_{-1},\\
x_{15n} &= \frac{1+z_{15n-1}}{y_{15n-2}} = \frac{1+z_{-1}}{\left(\frac{1+z_{-1}}{x_0}\right)} = x_0,\\
y_{15n} &= \frac{1+x_{15n-1}}{z_{15n-2}} = \frac{1+x_{-1}}{\left(\frac{1+x_{-1}}{y_0}\right)} = y_0,\\
y_{15n} &= \frac{1+y_{15n-1}}{x_{15n-2}} = \frac{1+y_{-1}}{\left(\frac{1+y_{-1}}{z_0}\right)} = z_0.
\end{aligned}
$$

$$
x_{15n+1} = \frac{1+z_{15n}}{y_{15n-1}} = \frac{1+z_0}{y_{-1}}, \quad y_{15n+1} = \frac{1+x_{15n}}{z_{15n-1}} = \frac{1+x_0}{z_{-1}},, \quad z_{15n+1} = \frac{1+y_{15n}}{x_{15n-1}} = \frac{1+y_0}{x_{-1}},
$$

also

$$
\begin{aligned}
x_{15n+2} &= \frac{1+z_{15n+1}}{y_{15n}} = \frac{1+\left(\frac{1+y_0}{x_{-1}}\right)}{y_0} = \frac{1+y_0+x_{-1}}{y_0x_{-1}},\\
y_{15n+2} &= \frac{1+x_{15n+1}}{z_{15n}} = \frac{1+\left(\frac{1+z_0}{y_{-1}}\right)}{z_0} = \frac{1+z_0+y_{-1}}{z_0y_{-1}},\\
z_{15n+2} &= \frac{1+y_{15n+1}}{x_{15n}} = \frac{1+\left(\frac{1+x_0}{z_{-1}}\right)}{x_0} = \frac{1+x_0+z_{-1}}{x_0z_{-1}}
\end{aligned}
$$

$$
\begin{aligned}
x_{15n+3} &= \frac{1+z_{15n+2}}{y_{15n+1}} = \frac{1+\left(\frac{1+x_0+z_{-1}}{x_0z_{-1}}\right)}{\frac{1+x_0}{z_{-1}}} = \frac{x_0\,(z_{-1}+1)+(1+z_{-1})}{x_0\,(1+x_0)} = \frac{1+z_{-1}}{x_0},\\
y_{15n+3} &= \frac{1+x_{15n+2}}{z_{15n+1}} = \frac{1+\left(\frac{1+y_0+x_{-1}}{y_0x_{-1}}\right)}{\frac{1+y_0}{x_{-1}}} = \frac{y_0\,(x_{-1}+1)+(1+x_{-1})}{y_0\,(1+y_0)} = \frac{1+x_{-1}}{y_0},\\
z_{15n+3} &= \frac{1+y_{15n+2}}{x_{15n+1}} = \frac{1+\left(\frac{1+z_0+y_{-1}}{z_0y_{-1}}\right)}{\frac{1+z_0}{y_{-1}}} = \frac{z_0\,(y_{-1}+1)+(1+y_{-1})}{z_0\,(1+z_0)} = \frac{1+y_{-1}}{z_0}.
\end{aligned}
$$

Identically, other relations may easily be proven. The proof is done.
The following theorem can be proves similarly.

3 Second System: $x_{n+1}=\dfrac{1-z_n}{y_{n-1}}, y_{n+1}=\dfrac{1-x_n}{z_{n-1}}, z_{n+1}=\dfrac{1-y_n}{x_{n-1}}$.

In this part, the solutions of the system of two difference equations have been examined

$$x_{n+1}=\frac{1-z_n}{y_{n-1}},\quad y_{n+1}=\frac{1-x_n}{z_{n-1}},\quad z_{n+1}=\frac{1-y_n}{x_{n-1}},\qquad n=0,1,..., \tag{2}$$

with a nonzero real numbers initial conditions.

Theorem 2 *Suppose that* $\{x_n, y_{n,} z_n\}$ *are solutions of system (2). Then*

1. $x_{n+5}=y_n,\ y_{n+5}=z_n,\ z_{n+5}=x_n$ for $n\geq -1$.
2.$x_{n+10}=z_n,\ y_{n+10}=x_n,\ z_{n+10}=y_n$ for $n\geq -1$.
3. $\{x_n\}_{n=-1}^{+\infty}$ and $\{y_n\}_{n=-1}^{+\infty}$ and $\{z_n\}_{n=-1}^{+\infty}$ are periodic with period ten. i.e., $x_{n+15}=x_n,\ y_{n+15}=y_n,\ z_{n+15}=z_n$ for $n\geq -1$ and the solutions takes the form

$$\begin{aligned}
x_{10n-1} &= x_{-1},\quad x_{10n}=x_0,\quad x_{10n+1}=\frac{1-z_0}{y_{-1}},\quad x_{10n+2}=\frac{-1+y_0+x_{-1}}{y_0x_{-1}},\\
x_{10n+3} &= \frac{1-z_{-1}}{x_0},\quad x_{10n+4}=y_{_{-1}},\quad x_{10n+5}=y_0,\quad x_{10n+6}=\frac{1-x_0}{z_{-1}},\\
x_{10n+7} &= \frac{-1+z_0+y_{-1}}{z_0y_{-1}},\quad x_{10n+8}=\frac{1-x_{-1}}{y_0},\quad x_{10n+9}=z_{_{-1}},\quad x_{10n+10}=z_{0,}\\
x_{10n+11} &= \frac{1-y_0}{x_{-1}},\quad x_{10n+12}=\frac{-1+x_0+z_{-1}}{x_0z_{-1}},\quad x_{10n+13}=\frac{1-y_{-1}}{z_0}.
\end{aligned}$$

$$\begin{aligned}
y_{10n-1} &= y_{-1},\quad y_{10n}=y_0,\quad y_{10n+1}=\frac{1-x_0}{z_{-1}},\quad y_{10n+2}=\frac{-1+z_0+y_{-1}}{z_0y_{-1}},\\
y_{10n+3} &= \frac{1-x_{-1}}{y_0},\quad y_{10n+4}=z_{_{-1}},\quad y_{10n+5}=z_0,\quad y_{10n+6}=\frac{1-y_0}{x_{-1}},\\
y_{10n+7} &= \frac{-1+x_0+z_{-1}}{x_0z_{-1}},\quad y_{10n+8}=\frac{1-y_{-1}}{z_0},\quad y_{10n+9}=x_{_{-1}},\quad y_{10n+10}=x_{0,}\\
y_{10n+11} &= \frac{1-z_0}{y_{-1}},\quad y_{10n+12}=\frac{-1+y_0+x_{-1}}{y_0x_{-1}},\quad y_{10n+13}=\frac{1-z_{-1}}{x_0}.
\end{aligned}$$

and

$$
\begin{aligned}
z_{10n-1} &= z_{-1}, \quad z_{10n} = z_0, \quad z_{10n+1} = \frac{1-y_0}{x_{-1}}, \quad z_{10n+2} = \frac{-1+x_0+z_{-1}}{x_0 z_{-1}}, \\
z_{10n+3} &= \frac{1-y_{-1}}{z_0}, \quad z_{10n+4} = x_{-1}, \quad z_{10n+5} = x_0, \quad z_{10n+6} = \frac{1-z_0}{y_{-1}}, \\
z_{10n+7} &= \frac{-1+y_0+x_{-1}}{y_0 x_{-1}}, \quad z_{10n+8} = \frac{1-z_{-1}}{x_0}, \quad z_{10n+9} = y_{-1}, \quad z_{10n+10} = y_0, \\
z_{10n+11} &= \frac{1-x_0}{z_{-1}}, \quad z_{10n+12} = \frac{-1+z_0+y_{-1}}{z_0 y_{-1}}, \quad z_{10n+13} = \frac{1-x_{-1}}{y_0}.
\end{aligned}
$$

Or equivalently

$$
\begin{aligned}
\{x_n\}_{n=-1}^{+\infty} &= \left\{ \begin{array}{c} x_{-1}, x_0, \frac{1-z_0}{y_{-1}}, \frac{-1+y_0+x_{-1}}{y_0 x_{-1}}, \frac{1-z_{-1}}{x_0}, y_{-1}, y_0, \frac{1-x_0}{z_{-1}}, \frac{-1+z_0+y_{-1}}{z_0 y_{-1}}, \\ \frac{1-x_{-1}}{y_0}, z_{-1}, z_0, \frac{1-y_0}{x_{-1}}, \frac{-1+x_0+z_{-1}}{x_0 z_{-1}}, \frac{1-y_{-1}}{z_0}, x_{-1}, x_0, \ldots \end{array} \right\}, \\
\{y_n\}_{n=-1}^{+\infty} &= \left\{ \begin{array}{c} y_{-1}, y_{-1}, \frac{1-x_0}{z_{-1}}, \frac{-1+z_0+y_{-1}}{z_0 y_{-1}}, \frac{1-x_{-1}}{y_0}, z_{-1}, z_0, \frac{1-y_0}{x_{-1}}, \frac{-1+x_0+z_{-1}}{x_0 z_{-1}}, \\ \frac{1-y_{-1}}{z_0}, x_{-1}, x_0, \frac{1-z_0}{y_{-1}}, \frac{-1+y_0+x_{-1}}{y_0 x_{-1}}, \frac{1-z_{-1}}{x_0}, y_{-1}, y_0, \ldots \end{array} \right\}, \\
\{z_n\}_{n=-1}^{+\infty} &= \left\{ \begin{array}{c} z_{-1}, z_0, \frac{1-y_0}{x_{-1}}, \frac{-1+x_0+z_{-1}}{x_0 z_{-1}}, \frac{1-y_{-1}}{z_0}, x_{-1}, x_0, \frac{1-z_0}{y_{-1}}, \frac{-1+y_0+x_{-1}}{y_0 x_{-1}}, \\ \frac{1-z_{-1}}{x_0}, y_{-1}, y_0, \frac{1-x_0}{z_{-1}}, \frac{-1+z_0+y_{-1}}{z_0 y_{-1}} \frac{1-x_{-1}}{y_0}, z_{-1}, z_0, \ldots \end{array} \right\}.
\end{aligned}
$$

4 Numerical Examples

Significant numerical examples have been considered in this part in order to show the results obtained earlier, and also to enhance our theoretical discussion.
Example 1. Consider the difference system equation (1) with the initial conditions $x_{-1} = 0.2$, $x_0 = 0.6$, $y_{-1} = 3$, $y_0 = 7$, $z_{-1} = -5$ *and* $z_0 = -1.6$. (See Fig. 1).

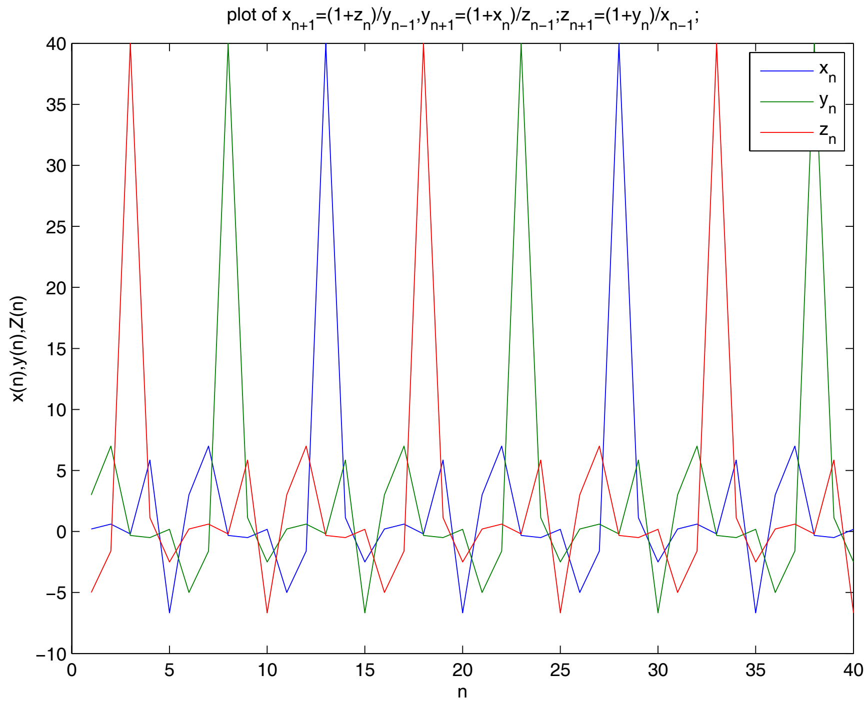

Figure 1. This figure shows the solutions of the system

$$x_{n+1} = \frac{1+z_n}{y_{n-1}}, \quad y_{n+1} = \frac{1+x_n}{z_{n-1}}, \quad z_{n+1} = \frac{1+y_n}{x_{n-1}},$$

where $x_{-1} = 0.2$, $x_0 = 0.6$, $y_{-1} = 3$, $y_0 = 7$, $z_{-1} = -5$ *and* $z_0 = -1.6$.

Example 2. For the initial conditions $x_{-1} = 9$, $x_0 = -0.2$, $y_{-1} = 2$, $y_0 = 0.7$, $z_{-1} = 6$ *and* $z_0 = 1.6$. when we take the system (1). (See Fig. 2).

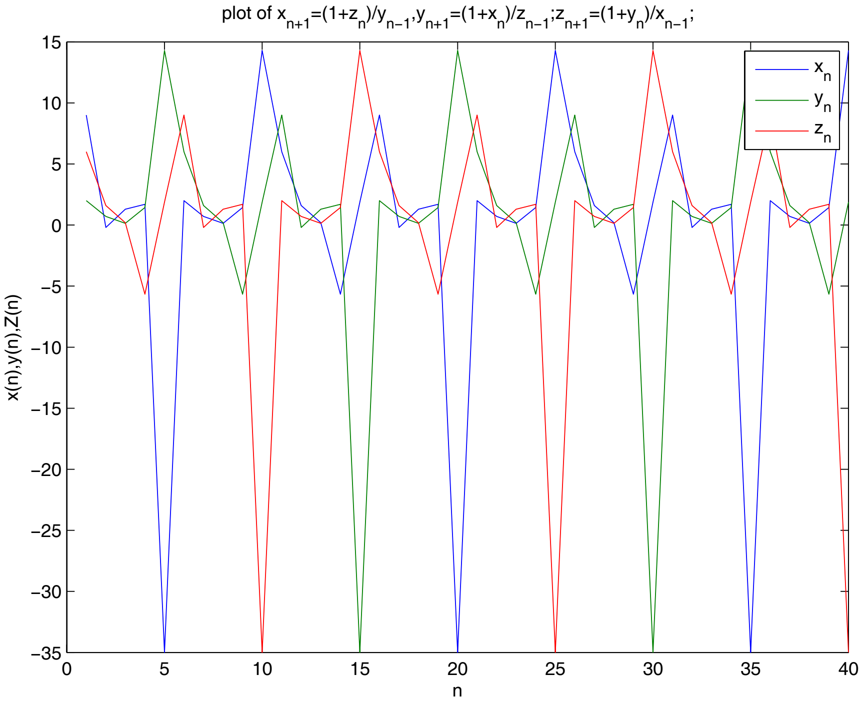

Figure 2. This figure shows the periodicity of system (1)
where $x_{-1} = 9$, $x_0 = -0.2$, $y_{-1} = 2$, $y_0 = 0.7$, $z_{-1} = 6$ *and* $z_0 = 1.6$.

Example 3. Take the initial conditions as follows $x_{-1} = 9$, $x_0 = -0.2$, $y_{-1} = 2$, $y_0 = 0.7$, $z_{-1} = 6$ *and* $z_0 = 1.6$ for the difference system equation (2). See Fig. 3.

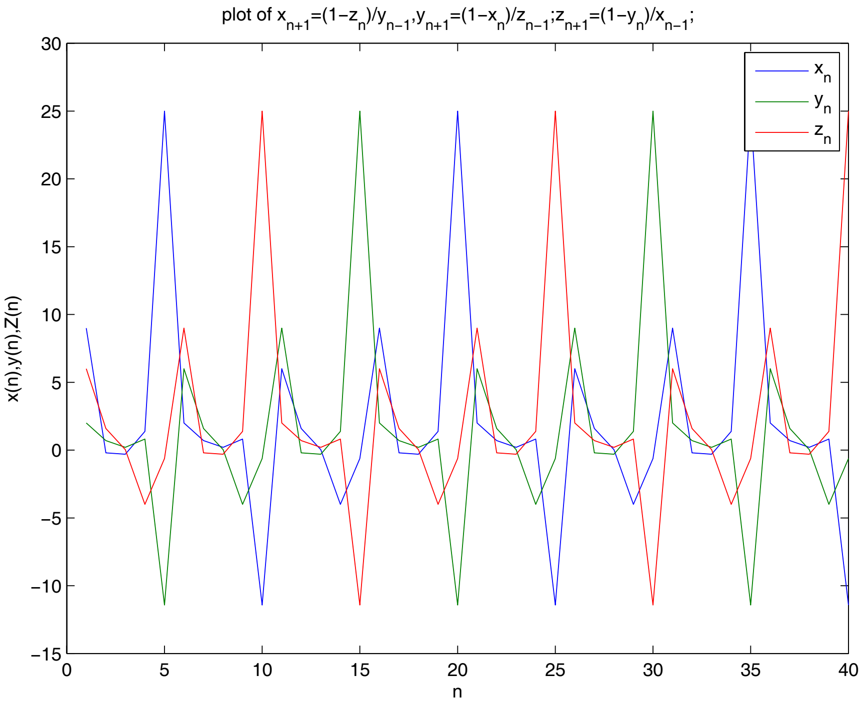

Figure 3. This figure shows the periodicity of the solutions of the system
$$x_{n+1}=\frac{1-z_n}{y_{n-1}},\quad y_{n+1}=\frac{1-x_n}{z_{n-1}},\quad z_{n+1}=\frac{1-y_n}{x_{n-1}},$$
where $x_{-1}=9$, $x_0=-0.2$, $y_{-1}=2$, $y_0=0.7$, $z_{-1}=6$ *and* $z_0=1.6$.

Example 4. Put the initial conditions $x_{-1}=9$, $x_0=2$, $y_{-1}=0.2$, $y_0=-3$, $z_{-1}=-1.4$ *and* $z_0=4$ in the system (2). (See Fig. 4).

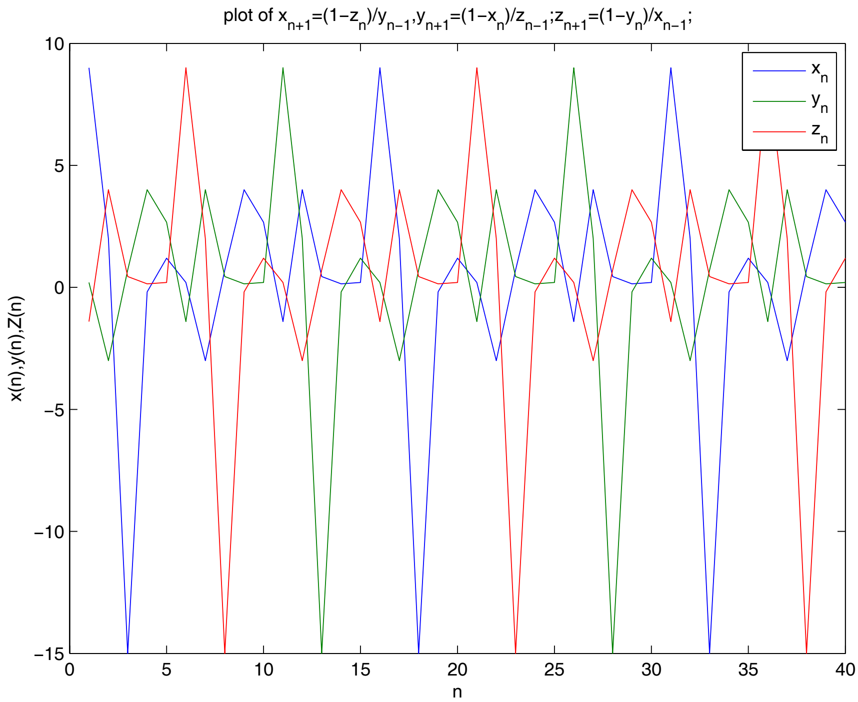

Figure 4. This figure shows the solutions of the system (2), where $x_{-1} = 9$, $x_0 = 2$, $y_{-1} = 0.2$, $y_0 = -3$, $z_{-1} = -1.4$ *and* $z_0 = 4$.

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