

A unique Solution of Stochastic Partial Differential Equations with Non-Local Initial condition

Mahmoud M. El-Borai

Department of Mathematics, Faculty of Science, Alexandria University,
Alexandria, Egypt
m_m_elborai@yahoo.com

A. Tarek S.A.

Department of Mathematics, Faculty of Science, Alexandria University,
Alexandria, Egypt
ahmedsayed2890@gmail.com

Abstract

In this paper, we shall discuss the uniqueness "pathwise uniqueness" of the solutions of stochastic partial differential equations (SPDEs) with non-local initial condition,

$$du(x, t) = \sum_{|q| \leq 2m} a_q(x, t) D^q u(x, t) dt + b(u(x, t)) dt + \sigma(u(x, t)) dB(t)$$

$$u(x, 0) = \phi(x) + \sum_{i=1}^p c_i u(x, t_i) \quad (1)$$

We shall use the Yamada-Watanabe condition for "pathwise uniqueness" of the solutions of the stochastic differential equation; this condition is weaker than the usual Lipschitz condition. The proof is based on Bihari's inequality.

Keywords: Stochastic partial differential equation, Pathwise uniqueness, Bihari's inequality.

1 Introduction

Our main result is using the Yamada-Watanabe condition, which relaxes the Lipschitz condition for the pathwise uniqueness of the solutions of stochastic differential equation in [3],[4] in the proof the pathwise uniqueness of (1). Before starting the main theorem, we start with some definitions and theorems necessary for the sequel.

2 Materials and Methods

Definition 1. *The triple $(\Omega, \mathfrak{F}, \mathbb{P})$ consisting of a sample space Ω , the σ -algebra \mathfrak{F} of subsets of Ω and a probability measure \mathbb{P} defined on \mathfrak{F} is known as a probability space.*

Definition 2. *A filtration is a family $\{\mathfrak{F}_t\}_{(t>0)}$ of increasing sub- σ -algebra of \mathfrak{F} (i.e., $\mathfrak{F}_t \subset \mathfrak{F}_s \subset \mathfrak{F}$, $\forall 0 \leq t < s < \infty$).*



Remark 1. The probability space together with its family of increasing sub- σ -algebra denoted by $(\Omega, \mathfrak{F}, \mathfrak{F}_t, \mathbb{P})$ is called a standard filtration space.

Definition 3. Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a probability space. A real-valued function $X : \Omega \rightarrow R$ is called \mathfrak{F} -measurable or random variable, if for all $a \in R$, $\{\omega \in \Omega : X(\omega) \leq a\} \in \mathfrak{F}$.

Definition 4. A family of random variables $X_t, t \in I$, where $I \subset R$ is an interval defined on a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ and indexed by a parameter t takes all possible values of I is called a stochastic process.

Definition 5. Let $(\Omega, \mathfrak{F}, \mathfrak{F}_t, \mathbb{P})$ be a standard filtration space and $I \subset R$ be an interval. The stochastic process X_t is said to be \mathfrak{F}_t -adapted if for all $t \in I$, the random variable X_t is \mathfrak{F}_t -measurable.

We further define the expectation $\mathbb{E}[X] = \int_{\Omega} X d\mathbb{P}$, for any random variable X .

Theorem 1 (Bihari's inequality). Let I denote an interval of the real line of the form $[a, \infty), [a, b]$ or $[a, b]$ with $a < b$. Let $\beta, v : I \rightarrow [0, \infty)$ and $\gamma : [0, \infty) \rightarrow [0, \infty)$ be three functions, where v and γ are continuous on I , β is continuous on the interior of I with $\int_a^t \beta(s) ds < \infty$ for all $t \in I$ and γ is non-decreasing and strictly positive on $(0, \infty)$,

a. If, for some $\alpha > 0$, the function v satisfies the inequality

$$v(t) \leq \alpha + \int_a^t \beta(s) \gamma(v(s)) ds, \quad t \in I \quad (2)$$

then

$$v(t) \leq F^{-1} \left(\int_a^t \beta(s) ds \right), \quad t \in [a, T]$$

where F^{-1} is the inverse function of

$$F(x) = \int_a^x \frac{dy}{\gamma(y)}, \quad x > 0.$$

and $T = \sup\{t \in I \mid \int_a^t \beta(s) ds < \int_a^{\infty} \frac{dy}{\gamma(y)}\}$

b. If the function v satisfies (2) with $\alpha = 0$ and $\int_0^x \frac{dy}{\gamma(y)} = +\infty \quad \forall x > 0$ then $v(t) = 0 \quad t \in I$.

Proof. for proof see [5]. □

Definition 6. $\mathcal{L}^2(\Omega, \mathcal{H})$; collection of all strongly measurable \mathcal{H} -valued random variables is a banach space equipped with the norm $\|\bullet\|_{\mathcal{L}^2} := [\mathbb{E} \|\bullet\|_{\mathcal{H}}^2]^{1/2}$

Theorem 2. Consider the SPDE's

$$du(x, t) = \sum_{|q| \leq 2m} a_q(x, t) D^q u(x, t) dt + b(u(x, t)) dt + \sigma(u(x, t)) dB(t)$$

with non-local initial condition

$$u(x, 0) = \phi(x) + \sum_{i=1}^p c_i u(x, t_i)$$

where $x \in R^n$, $B(t)$ is a standard Brownian motion defined over the standard filtration space $(\Omega, \mathfrak{F}, \mathfrak{F}_t, \mathbb{P})$, $D = (D_1, \dots, D_n)$, $q = (q_1, \dots, q_n)$, $D_i := \frac{\partial}{\partial x_i}$, $D^q = D_1^{q_1}, \dots, D_n^{q_n}$ and q is a multi-index, $|q| = q_1 + \dots + q_n$, $0 \leq t_1 < \dots < t_p$.

Equation (1) is called parabolic in the region $\Gamma = \{(x, t) : x \in R^n, t \geq 0\}$, if for any point $(x, t) \in \Gamma$ the real part of the λ -roots of the characteristic equation

$$\text{Det} \left[(-1)^m \sum_{|q|=2m} a_q(x, t) \xi^q - \lambda \mathbf{I} \right] = 0,$$

satisfy the inequality $\text{Re}[\lambda(x, t, \xi)] \leq -\delta |\xi|^m$ where δ is a positive constant, $\xi \in R^n$, $\xi^q = \xi_1^{q_1} \dots \xi_n^{q_n}$, \mathbf{I} is the unit matrix. We suppose that the coefficients a_q , $|q| \leq 2m$ are continuous and bounded on R^{n+1} and satisfy the Hölder condition with respect to x . Under these conditions, there exists a fundamental solution $\Theta(x, t, y, \theta)$ which satisfies

1. $\frac{d\Theta}{dt} = \sum_{|q| \leq 2m} a_q(x, t) D^q \Theta(x, t, y, \theta)$, $t > 0$, $x, y \in R^n$.
2. $\frac{\partial \Theta}{\partial t}$ and $D^q \Theta \in \mathcal{C}(\Gamma_1)$ such that $\Gamma_1 = \{(x, t, y, \theta) \in R^{2n} \times (0, \infty) \times (0, \infty)\}$, $|q| \leq 2m$.
3. $\|D^q(x, t, y, \theta)\| \leq [\frac{A_1}{t^\zeta}] e^{-A_2 \zeta_1}$, $\zeta_1 = \sum_{i=1}^n |x_i - y_i|^{\frac{2m}{2m-1}} t^{\frac{-1}{2m-1}}$, $\zeta = -\frac{n+|q|}{2m}$ and A_1, A_2 are positive constants.

Definition 7. By a solution of the equation (1), we mean a family of stochastic processes $\Upsilon = \{u, B(t)\}$ defined on a standard filtration space $(\Omega, \mathfrak{F}, \mathfrak{F}_t, \mathbb{P})$ such that

1. With probability one, u and $B(t)$ are continuous in t and $B(0) = 0$.
2. They are adapted to \mathfrak{F}_t , i.e., for each t , u and $B(t)$ are \mathfrak{F}_t -measurable.
3. $B(t)$ is a system of \mathfrak{F}_t -martingale such that $\langle B^i, B^j \rangle = \delta_{ij} \cdot t$, $i, j = 1, 2, \dots, n$.
4. Theorem (2) holds.
5. $\Upsilon = \{u, B(t)\}$ satisfies

$$\begin{aligned} u(x, t) &= \int_{R^n} \Theta(x, t, y, 0) u(y, 0) dy \\ &+ \int_0^t \int_{R^n} \Theta(x, t, y, s) b(u(y, s)) dy ds \\ &+ \int_0^t \int_{R^n} \Theta(x, t, y, s) b(u(y, s)) dy dB(s). \end{aligned} \quad (3)$$

where the integral by $dB(s)$ is understood in the sense of the stochastic integral.

Definition 8 (Pathwise Uniqueness). We shall say that the pathwise (strong) uniqueness holds for (1) if, for any two solutions $\Upsilon = \{u, B(t)\}$ and $\tilde{\Upsilon} = \{\tilde{u}, \tilde{B}(t)\}$, defined on a same filtration space $(\Omega, \mathfrak{F}, \mathfrak{F}_t, \mathbb{P})$, $u(x, 0) = \tilde{u}(x, 0)$ and $B(t) \equiv \tilde{B}(t)$ imply $u \equiv \tilde{u}$.

It supposed that $cM^* < 1$ where $c = \sum_{i=1}^p |c_i|$.

Theorem 3. *If $u \in \mathcal{C}([0, T]; \mathcal{H})$ is an \mathfrak{F}_t -adapted stochastic process and satisfies equation (3), then $u(x, t)$ satisfies the following equation*

$$\begin{aligned} u(x, t) &= Z(t)\Lambda^{-1}\phi(x) + Z(t)\Lambda^{-1} \sum_{i=1}^p c_i \int_0^{t_i} Z(t_i)b(u(x, s))ds \\ &+ Z(t)\Lambda^{-1} \sum_{i=1}^p c_i \int_0^{t_i} Z(t_i)\sigma(u(x, s))dB(s) \\ &+ \int_0^t Z(t)b(u(x, t))ds + \int_0^t Z(t)\sigma(u(x, t))dB(s). \end{aligned} \quad (4)$$

where $\Lambda = I - \sum_{i=1}^p c_i Z(t_i)$ and $Z(t)$ is an operator defined as

$$Z(t)f = \int_{R^n} \Theta(x, t, y, 0)fdy$$

Proof.

$$\begin{aligned} \sum_{i=1}^p c_i u(x, t_i) &= \sum_{i=1}^p c_i \int_{R^n} \Theta(x, t_i, y, 0)[\phi(y) + \sum_{j=1}^p c_j u(y, t_j)] \\ &+ \sum_{i=1}^p c_i \int_0^{t_i} \int_{R^n} \Theta(x, t_i, y, s)b(u(y, s))dyds \\ &+ \sum_{i=1}^p c_i \int_0^{t_i} \int_{R^n} \Theta(x, t_i, y, s)\sigma(u(y, s))dydB(s) \\ \\ \sum_{i=1}^p c_i u(x, t_i) &- \sum_{i=1}^p c_i \sum_{j=1}^p c_j \int_{R^n} \Theta(x, t_i, y, 0)u(y, t_j) \\ &= \sum_{i=1}^p c_i \int_{R^n} \Theta(x, t_i, y, 0)\phi(y) \\ &+ \sum_{i=1}^p c_i \int_0^{t_i} \int_{R^n} \Theta(x, t_i, y, s)b(u(y, s))dyds \\ &+ \sum_{i=1}^p c_i \int_0^{t_i} \int_{R^n} \Theta(x, t_i, y, s)\sigma(u(y, s))dydB(s) \\ \\ \Lambda \sum_{i=1}^p c_i u(y, t_i) &= \sum_{i=1}^p c_i Z(t_i)\phi(y) + \sum_{i=1}^p c_i \int_0^{t_i} Z(t_i)b(u(x, s))ds \\ &+ \sum_{i=1}^p c_i \int_0^{t_i} Z(t_i)\sigma(u(x, s))dB(s) \end{aligned}$$

using $u(x, 0) = \phi(x) + \sum_{i=1}^p c_i u(x, t_i)$ and multiply with $Z(t)$,

$$\begin{aligned} Z(t)\phi(x) + Z(t) \sum_{i=1}^p c_i u(x, t_i) &= Z(t)\phi(x) + Z(t)\Lambda^{-1} \sum_{i=1}^p c_i Z(t_i)\phi(x) \\ &+ Z(t)\Lambda^{-1} \sum_{i=1}^p c_i \int_0^{t_i} Z(t_i)b(u(x, s))ds \\ &+ Z(t)\Lambda^{-1} \sum_{i=1}^p c_i \int_0^{t_i} Z(t_i)\sigma(u(x, s))dB(s), \end{aligned}$$

It is easy to see that $\Lambda^{-1} = I + \Lambda^{-1} \sum_{i=1}^p c_i Z(t_i)$, then we get the result. \square

3 Main Result

In this section, we state and discuss the main theorem for this paper.

Theorem 4. Let $\sigma(x) = \begin{bmatrix} \sigma_1(x_1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_n(x_n) \end{bmatrix}$, $b(x) = (b_1(x), \dots, b_n(x))$

such that:

1. There exists a positive increasing function $\rho(\varrho)$, $\varrho \in (0, \infty)$ such that

$$|\sigma_i(\tau) - \sigma_i(\eta)| \leq \rho(|\tau - \eta|), \quad \tau, \eta \in R, \quad i = 1, 2, \dots, n.$$

and

$$\int_{0+} \rho^{-2}(\varrho) d\varrho = +\infty$$

2. There exists a positive increasing concave function $\kappa(\varrho)$, $\varrho \in (0, \infty)$ such that

$$|b_i(x) - b_i(y)| \leq \kappa(\|x - y\|), \quad x, y \in R^n, \quad i = 1, 2, \dots, n.$$

and

$$\int_{0+} \kappa^{-1}(\varrho) d\varrho = +\infty$$

3. Theorem (2) holds.

then the pathwise uniqueness of the solutions holds for (1).

Proof. Let $a_0 = 1 > a_1 > a_2 > \dots > a_k \rightarrow 0$ be defined by

$$\int_{a_1}^{a_0} \rho^{-2}(\varrho) d\varrho = 1, \int_{a_2}^{a_1} \rho^{-2}(\varrho) d\varrho = 2, \dots, \int_{a_k}^{a_{k-1}} \rho^{-2}(\varrho) d\varrho = k, \dots.$$

then there exists a twice continuity differentiable function $\psi_k(\varrho)$ on $[0, \infty)$ such that $\psi_k(0) = 0$,

$$\psi'_k(\varrho) = \begin{cases} 0 & , \quad 0 \leq \varrho \leq a_k \\ \text{between } 0 \text{ and } 1 & , \quad a_k \leq \varrho \leq a_{k-1} \\ 1 & , \quad \varrho \geq a_{k-1} \end{cases}$$

and

$$\psi_k''(\varrho) = \begin{cases} 0 & , \quad 0 \leq \varrho \leq a_k \\ \text{between } 0 \text{ and } \frac{2}{k} \cdot \rho^{-2}(\varrho) & , \quad a_k \leq \varrho \leq a_{k-1} \\ 0 & , \quad \varrho \geq a_{k-1} \end{cases}$$

we extend $\psi_k(\varrho)$ on $(-\infty, \infty)$ symmetrically, i.e., $\psi_k(\varrho) = \psi_k(|\varrho|)$ clearly $\psi_k(\varrho)$ is a twice continuously differentiable function on $(-\infty, \infty)$ such that $\psi_k(\varrho) \uparrow |\varrho|$ as $k \rightarrow \infty$.

Now let $\{u, B(t)\}$ and $\{\bar{u}, \bar{B}(t)\}$ be two solutions of (1) on the same probability space such that $u(x, 0) = \bar{u}(x, 0)$ and $B(t) \equiv \bar{B}(t)$ then,

$$\begin{aligned} u^j(x, t) - \bar{u}^j(x, t) &= \int_0^t Z(t) [\sigma_j(u^j(y, s)) - \sigma_j(\bar{u}^j(y, s))] dB^j(s) \\ &+ Z(t)\Lambda^{-1} \sum_{i=1}^p c_i \int_0^{t_i} Z(t_i) [\sigma_j(u^j(y, s)) - \sigma_j(\bar{u}^j(y, s))] dB^j(s) \\ &+ Z(t)\Lambda^{-1} \sum_{i=1}^p c_i \int_0^{t_i} Z(t_i) [b_j(u(y, s)) - b_j(\bar{u}(y, s))] ds \\ &+ \int_0^t Z(t) [b_j(u(y, s)) - b_j(\bar{u}(y, s))] ds \end{aligned}$$

According to theorem (3), there is a positive constant M such that $\|Z(t)\|_{\mathcal{H}} \leq M$, and by Ito's formula,

$$\begin{aligned} \psi_k(u(x, t) - \bar{u}(x, t)) &= \int_0^t \psi_k'(u^j - \bar{u}^j) Z(t) [\sigma_j(u^j(y, s)) - \sigma_j(\bar{u}^j(y, s))] dB^j(s) \\ &+ \int_0^t \psi_k'(u^j - \bar{u}^j) \left[Z(t)\Lambda^{-1} \sum_{i=1}^p c_i \int_0^{t_i} Z(t_i) \{\sigma_j(u^j(y, s)) - \sigma_j(\bar{u}^j(y, s))\} \right] dB^j(s) \\ &+ \int_0^t \psi_k'(u^j - \bar{u}^j) \left[Z(t)\Lambda^{-1} \sum_{i=1}^p c_i \int_0^{t_i} Z(t_i) \{b_j(u(y, s)) - b_j(\bar{u}(y, s))\} \right] ds \\ &+ 1/2 \int_0^t \psi_k''(u^j - \bar{u}^j) \left[Z(t)\Lambda^{-1} \sum_{i=1}^p c_i \int_0^{t_i} Z(t_i) \{\sigma_j(u^j(y, s)) - \sigma_j(\bar{u}^j(y, s))\} \right]^2 ds \\ &+ \int_0^t \psi_k'(u^j - \bar{u}^j) [Z(t) \{b_j(u(y, s)) - b_j(\bar{u}(y, s))\}] ds \\ &+ 1/2 \int_0^t \psi_k''(u^j - \bar{u}^j) [Z(t) \{\sigma_j(u^j(y, s)) - \sigma_j(\bar{u}^j(y, s))\}]^2 ds \\ &= I_1 + I_2 + I_3 + I_4 + I_5 + I_6 \end{aligned}$$

It is clear that $\mathbb{E}[I_1] = \mathbb{E}[I_2] = 0$ and since ψ_k' is uniformly bounded, κ is concave

$$\begin{aligned} \|\mathbb{E}[I_5]\| &\leq k_1 \int_0^t \mathbb{E}[\kappa(\|u - \bar{u}\|)] ds \\ &\leq k_1 \int_0^t \kappa(\mathbb{E}\|u - \bar{u}\|) ds \end{aligned}$$

by Jensen's inequality. Similarly for I_3 .
We have, for I_6

$$\begin{aligned} \| I_6 \| &\leq 1/2 \int_0^t \psi_k''(u^j - \bar{u}^j) \| Z(t) \|^2 \rho^2(|u^i - \bar{u}^i|) ds \\ &\leq k_2 \cdot t \max_{a_k \leq |\varrho| \leq a_{k-1}} [\psi_k''(\varrho) \rho^2(\varrho)] \\ &\leq k_2 \cdot t \cdot \frac{2}{k} \rightarrow 0 \quad \text{as } k \rightarrow \infty \end{aligned}$$

Similarly for I_4 .

Where k_1 and k_2 are positive constants. Also, $\psi_k(u^i - \bar{u}^i) \uparrow |u^i - \bar{u}^i|$ as $k \rightarrow \infty$,

$$\mathbb{E}(|u^i - \bar{u}^i|) \leq k_1 \int_0^t \kappa(\mathbb{E} \|u - \bar{u}\|) ds, \quad i = 1, 2, \dots, n$$

and hence, we have

$$\mathbb{E}(\|u - \bar{u}\|) \leq k_3 \int_0^t \kappa(\mathbb{E} \|u - \bar{u}\|) ds,$$

where k_3 is positive constant.

By using theorem (2), this implies $\mathbb{E}(\|u - \bar{u}\|) = 0$ and therefore $u \equiv \bar{u}$ \square

References

- [1] R.Durrett, *Stochastic Calculus: A practical Introduction*, (1996).
- [2] Brent Oksendal, *Stochastic Differential Equations*, Fifth Edition, Springer-Verlag.
- [3] T. YAMADA and S. WATANABE, *On the uniqueness of solutions of stochastic differential equations*, J. Math. Kyoto Univ, 155-167, (1971) .
- [4] T. YAMADA and S. WATANABE, *On the uniqueness of solutions of stochastic differential equations II*, J. Math. Kyoto Univ, 553-563, (1971) .
- [5] S. ALTAY and UWE SCHMACK, *Lecture notes on the Yamada- Watanabe Condition for the pathwise uniqueness of solutions of certain stochastic Differential Equations*, (2013).
- [6] Mahmoud M. El-Borai and Farouk K. Assaad, *On the asymptotic Behavior of Some Nonlinear Parabolic Systems*, KYUNGPOOK Math. J, 37-42, 37(1997).
- [7] Mahmoud M. El-Borai, *On some fractional evolution equations with non-local conditions*, International J. of pure and appl. Math. , vol 24, No. 3, 405-415, (2005).
- [8] Mahmoud M. El-Borai, *On some stochastic fractional integro-differential equations*, *Advanced in dynamical system and application*, vol.1, No.1, 49-57, (2006).

- [9] Mahmoud M.El-Borai, Khairia El-Said El-Nadi, Osama L. Mostafa and hamdy M. Ahmed, *Volterra Equations with fractional stochastic integrals*, Mathematical problems in Engineering, 453-468, 5 (2004).
- [10] K. Balachandran, S. Kiruthika, *Existence results for fractional integro-differential equations with nonlocal condition via resolvent operators*, Comput. Math. Appl, 1350-1358, 62 (2011).