

CONGRUENCES ON *-SIMPLE TYPE A I-SEMIGROUPS¹Ndubuisi R.U, ²Asibong-Ibe U.I, ³Udoaka O.G^{1,2}Department of Mathematics & Statistics, University of Port Harcourt, Port Harcourt, Nigeria.³Department of Mathematics & Statistics, Akwa Ibom State University, Ikot Akpaden, Nigeria.

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Abstract

This paper obtains a characterisation of the congruences on *-simple type A I-semigroups. The *-locally idempotent-separating congruences, strictly *-locally idempotent-separating congruences and minimum cancellative monoid congruences, are characterised.

Keywords: Type A I-Semigroup, *-Locally Idempotent-Separating, Cancellative Monoid Congruence, Generalized Bruck-Reilly *-Extensions.

1. Introduction

For a semigroup S , $E(S)$ will denote the set of idempotents of S . If S is a semigroup with non-empty set of idempotents $E(S)$, we define a partial order " \leq " on $E(S)$ such that $e \leq f$ if and only if $ef = fe = e$. Let I denote the set of all integers and let \mathbb{N}^0 denote the set of non-negative integers. A semigroup S is said to be an I -semigroup if and only if $E(S)$ is order isomorphic to I under the reverse of the partial order.

The structure theorem for *-simple type A I-semigroups was established in [8], as an extension of the structure theorem for simple I-inverse semigroups and *-simple type A ω -semigroups due to Warne [10] and Asibong-Ibe [1]. This paper is a follow up of the study of congruences on *-bisimple type A I-semigroups studied by Ndubuisi and Asibong-Ibe [7], where the congruences were identified as idempotent-separating congruence and minimum cancellative monoid congruence.

Earlier investigations in [6] and [10] studied congruences on *-simple type A ω -semigroups and congruences on simple I-inverse semigroups respectively. Determination of congruences throughout this paper is based on their description in [6].

This work is divided as follows. Section 2 contains a minimum of results concerning *-simple type A I -semigroups. The content of section 3 is a determination of *-locally idempotent-separating congruences, strictly *-locally idempotent-separating congruences and minimum cancellative monoid congruences of a *-simple type A I-semigroup.

Let us recall some definitions which will be useful in the study.

Let S be a semigroup and let $a, b \in S$. Then the elements a and b are said to be \mathcal{R}^* -related written $a \mathcal{R}^* b$ if and only if for all $x, y \in S^1$, $xa = ya$ if and only if $xb = yb$. The relation \mathcal{L}^* is defined

dually. The join of the equivalence relations \mathcal{R}^* and \mathcal{L}^* is denoted by \mathcal{D}^* and their intersection by \mathcal{H}^* . Thus $a \mathcal{H}^* b$ if and only if $a \mathcal{R}^* b$ and $a \mathcal{L}^* b$. In general $\mathcal{R}^* \circ \mathcal{L}^* \neq \mathcal{L}^* \circ \mathcal{R}^*$ as shown in [3].

Following Fountain [4] a semigroup is an abundant semigroup if every \mathcal{L}^* -class and every \mathcal{R}^* -class in S contain idempotents. An abundant semigroup S is adequate [3] if $E(S)$ forms a semilattice. In an adequate semigroup every \mathcal{L}^* -class \mathcal{R}^* -class contains a unique idempotent.

Let a be an element of an adequate semigroup S , and a^* (a^\dagger) denotes the unique idempotent in the \mathcal{L}^* -class L_a^* (\mathcal{R}^* -class R_a^*) containing a .

We remark that a type A (in particular, right type A) semigroup realized in Fountain [2] as a special type of right PP monoid with e -cancellable element where $e \in E(S)$, the set of idempotents in S . An adequate semigroup S is said to be a type A semigroup if $ea = a(ea)^*$ and $ae = (ae)^\dagger a$ for all $a \in S$ and $e \in E(S)$.

We conclude this section by defining the relation \mathcal{J}^* . Let S be a semigroup and I^* be an ideal of S . Then I^* is said to be a $*$ -ideal if $L_a^* \subseteq I^*$ and $R_a^* \subseteq I^*$ for all $a \in I^*$. The smallest $*$ -ideal containing 'a' is the principal $*$ -ideal generated by 'a' and is denoted by $J^*(a)$. For $a, b \in S$, $a \mathcal{J}^* b$ if and only if $J^*(a) = J^*(b)$. The relations \mathcal{J}^* contains \mathcal{D}^* .

A semigroup S is said to be $*$ -simple if the only $*$ -ideal of S is itself. Clearly a semigroup is $*$ -simple if all its elements are \mathcal{J}^* -related. To have a clear picture of \mathcal{J}^* -related elements we recall the following Lemma.

Lemma 1.1 [3]. Let S be a semigroup and $a, b \in S$. Then $b \in J^*(a)$ if and only if there are elements $a_0, a_1, \dots, a_n \in S$, $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n \in S^1$ such that $a = a_0$, $b = a_n$ and $a_i \mathcal{D}^* x_i a_{i-1} y_i$, for $i = 1, 2, \dots, n$.

Other basic results discussed in [3] will be assumed. The notation adopted in this paper is similar to that in Fountain [3], Howie [5], Asibong-Ibe [1] and Makanjuola [6].

Recently type A semigroups have been shown to be special type of restriction semigroups. In this case type A ω -semigroup will essentially be an ω -restriction semigroups. The idea developed here will prove useful in the study of restriction semigroups.

However, we will in this work retain the term type A semigroups generally.

2. $*$ -Simple Type A I-Semigroups

Following [9], let $T = \cup_{i=0}^{d-1} M_i$ be a chain of cancellative monoids. Each element $x_i \in T$ is necessarily in M_i for $0 \leq i \leq d - 1$. An identity $e_i \in M_i$ is an idempotent in T . Clearly $e_i \in T$ form a chain of idempotents $e_0 > e_1 > \dots > e_{d-1}$.

Let $\theta : T \rightarrow M_0$ be a monoid morphism and let $S = T \times I \times I$ (where I is the set of all integers) be the set of all ordered triples (x_i, m, n) where $m \in \mathbb{N}^0, n \in I, 0 \leq i \leq d - 1$ and $x_i \in T$.

Define multiplication on S by the rule

$$(x_i, m, n)(y_j, p, q) = \begin{cases} (x_i \cdot f_{n-p,p}^{-1} \cdot y_j \theta^{n-p} \cdot f_{n-p,q}, m, n + q - p) & \text{if } n \geq p \\ (f_{p-n,m}^{-1} \cdot x_i \theta^{p-n} \cdot f_{p-n,n}, y_j, m + p - n, q) & \text{if } n \leq p \end{cases}$$

where θ^0 is the identity automorphism of T , and for $m \in \mathbb{N}^0, n \in I, f_{0,n} = e_i$, the identity of M_i , while for $m > 0, f_{m,n} = u_{n+1} \theta^{m-1} \cdot u_{n+2} \theta^{m-2} \dots u_{n+(m-1)} \theta \cdot u_{n+m}$, and

$f_{m,n}^{-1} = u_{n+m}^{-1} \cdot u_{n+(m-1)}^{-1} \theta \dots u_{n+2}^{-1} \theta^{m-2} \cdot u_{n+1}^{-1} \theta^{m-1}$, where $\{u_n : n \in I\}$ is a collection of T with $u_n = e_i$ for $n > 0$. Denote a semigroup formed by $S = GBR^*(T, \theta)$ where $T = \cup_{i=0}^{d-1} M_i$.

If for each i we now let $M_i = \{e_i\}$, a monoid with one element, we obtain the set $I \times I$ under the multiplication

$$(md + i, nd + i)(pd + j, qd + j) = \begin{cases} ((md + i, (n + q - p)d + i) & \text{if } n \geq p \\ ((m + p - n)d + j, qd + j) & \text{if } n \leq p \end{cases}$$

We denote $I \times I$ under the above multiplication by B_d^* and call it the extended bicyclic semigroup.

If we let (x_i, m, n) be an idempotent in S . Then

$$(x_i, m, n) = (x_i, m, n)(x_i, m, n) = \begin{cases} (x_i \cdot f_{n-m, m}^{-1} \cdot x_i \theta^{n-m} \cdot f_{n-m, n}, m, n - m + n) & \text{if } n \geq m \\ (f_{m-n, m}^{-1} \cdot x_i \theta^{m-n} \cdot f_{m-n, n} \cdot x_i, m - n + m, n) & \text{if } n \leq m \end{cases}$$

in which case $m = n$, $x_i^2 = x_i$.

Conversely, suppose $x_i^2 = x_i$ then we have that $(x_i, m, n)(x_i, m, n) = (x_i, m, n)$. Thus (x_i, m, n) is an idempotent if and only if $m = n$ and x_i is an idempotent in S .

The following results were proved in [8].

Lemma 2.1. Let $S = GBR^*(T, \theta)$ be a generalized Bruck-Reilly $*$ -extension of a monoid T , where $T = \bigcup_{i=0}^{d-1} M_i$ is a finite chain of cancellative monoids M_i . Let $(x_i, m, n), (y_j, p, q) \in S$. Then

- i) $(x_i, m, n) \mathcal{R}^*(y_j, p, q)$ if and only if $m = p$ and $i = j$.
- ii) $(x_i, m, n) \mathcal{L}^*(y_j, p, q)$ if and only if $n = q$ and $i = j$.
- iii) $(x_i, m, n) \mathcal{J}^*(y_j, p, q)$. That is S is $*$ -simple.

Lemma 2.2. $S = GBR^*(T, \theta)$ is a type A semigroup if and only if T is a type A semigroup

Theorem 2.3. Let $S = GBR^*(T, \theta)$ be the generalized Bruck-Reilly $*$ -extension of the monoid T where $T = \bigcup_{i=0}^{d-1} M_i$. Then S is a $*$ -simple type A I-semigroup with d \mathcal{D}^* -classes.

We conclude this section, with the structure theorem of $*$ -simple type A I-semigroups.

Theorem 2.4 [8]. Let S be a $*$ -simple type A I-semigroup with d \mathcal{D}^* -classes. Then S is isomorphic to a generalized Bruck-Reilly $*$ -extension $S = GBR^*(T, \theta)$ of a monoid T , where $T = \bigcup_{i=0}^{d-1} M_i$ is a finite chain of cancellative monoids M_i and θ is an endomorphism of T with image in M_0 .

3. The Congruences

In this section, we will determine the congruence relations on a $*$ -simple type A I-semigroup $S = GBR^*(T, \theta)$. We first present the properties of the congruences and then show that every congruence relation ρ on S is either a $*$ -locally idempotent-separating congruence (if no two distinct \mathcal{D}^* -related idempotents are ρ -related) or all the idempotents are in one ρ -class. We also provide a method for constructing the strictly $*$ -locally idempotent separating congruences. Lastly, we show that there is a minimum cancellative monoid congruence on S .

Lemma 3.1. Let $S = GBR^*(T, \theta)$ be a $*$ -simple type A I-semigroup where $T = \bigcup_{i=0}^{d-1} M_i$ is a semilattice of cancellative monoids. Then \mathcal{H}^* is a congruence on S , and $S/\mathcal{H}^* \cong B_d^*$.

Proof. The mapping $\theta : S \rightarrow B_d^*$ by

$$(x_i, m, n)\theta = (md + i, nd + i)$$

is onto. It is a homomorphism since

$$\begin{aligned}
 (x_i, m, n)(y_j, p, q)\theta &= \begin{cases} (x_i \cdot f_{n-p,p}^{-1} \cdot y_j \theta^{n-p} \cdot f_{n-p,q}, m, n + q - p) & \text{if } n \geq p \\ (f_{p-n,m}^{-1} \cdot x_i \theta^{p-n} \cdot f_{p-n,n} \cdot y_j, m + p - n, q) & \text{if } n \leq p \end{cases} \times \theta \\
 &= \begin{cases} (x_i \cdot f_{n-p,p}^{-1} \cdot y_j \theta^{n-p} \cdot f_{n-p,q}, m, n + q - p)\theta & \text{if } n \geq p \\ (f_{p-n,m}^{-1} \cdot x_i \theta^{p-n} \cdot f_{p-n,n} \cdot y_j, m + p - n, q)\theta & \text{if } n \leq p \end{cases} \\
 &= \begin{cases} (md + i, (n + q - p)d + i) & \text{if } n \geq p \\ ((m + p - n)d + j, qd + j) & \text{if } n \leq p \end{cases} \\
 &= (md + i, nd + i)(pd + j, qd + j) \\
 &= (x_i, m, n)\theta (y_j, p, q)\theta.
 \end{aligned}$$

Thus θ is a homomorphism.

Furthermore, $(x_i, m, n)(y_j, p, q) \in \mathcal{H}^*$ if and only if $(md + i, nd + i) = (pd + j, qd + j)$; hence $\theta \circ \theta^{-1} = \mathcal{H}^*$ and the result follows.

Lemma 3.2. Let ρ be a congruence on a $*$ -simple type A I -semigroup $S = GBR^*(T, \theta)$ where $T = \cup_{i=0}^{d-1} M_i$. Suppose that

- (i) $(e_i, m, m) \rho (e_j, m, m)$ then for any $n \in I$, $(e_i, n, n) \rho (e_j, n, n)$
- (ii) $(e_i, m, m) \rho (e_j, m + 1, m + 1)$ then for any $n \in I$, $(e_i, n, n) \rho (e_j, n + 1, n + 1)$

Proof. i) Let $(e_0, n, m), (e_0, m, n) \in S$, then

$$\begin{aligned}
 (e_0, n, m)(e_i, m, m)(e_0, m, n) &= (e_i, n, n). \\
 (e_0, n, m)(e_j, m, m)(e_0, m, n) &= (e_j, n, n),
 \end{aligned}$$

and

$$\begin{aligned}
 (e_0, m, n)(e_i, n, n)(e_0, n, m) &= (e_i, m, m). \\
 (e_0, m, n)(e_j, n, n)(e_0, n, m) &= (e_j, m, m).
 \end{aligned}$$

ii) Let $(e_0, n, m), (e_0, m, n) \in S$, then we have

$$\begin{aligned}
 (e_0, n, m)(e_i, m, n) &= (e_i, n, n). \\
 (e_0, n, m)(e_j, m + 1, m + 1)(e_0, m, n) &= (e_j, n + 1, n + 1),
 \end{aligned}$$

and

$$\begin{aligned}
 (e_0, m, n)(e_i, n, n)(e_0, n, m) &= (e_i, m, m). \\
 (e_0, m, n)(e_j, n + 1, n + 1)(e_0, n, m) &= (e_j, m + 1, m + 1).
 \end{aligned}$$

Hence the proof.

We now establish an important property of congruences on $*$ -simple type A I -semigroups.

Theorem 3.3. A congruence ρ on a $*$ -simple type A I -semigroup is either a $*$ -locally idempotent-separating congruence or all the idempotents are in one ρ -class.

Proof. Suppose that the idempotent elements of S are not in one ρ -class and $e_{md+i} \rho e_{(m+k)d+i}$ for some $m, k \in I, k > 0$ and $0 \leq i \leq d - 1$. We are to show that no two distinct \mathcal{D}^* -related idempotents are ρ -related. Let $k = 1$ which implies that $e_{md+i} \rho e_{(m+1)d+i}$. Using Lemma 3.2 and the fact that $e_m \rho e_n$ implies $e_m \rho e_k$ for every $n \leq k \leq m$ together with the transitive property of the congruence, we see that the idempotents are in one ρ -class which is contrary to our assumption. Thus, no two distinct \mathcal{D}^* -related idempotents are ρ -related. Therefore ρ is a $*$ -locally idempotent-separating congruence. This completes the proof.

A typical $*$ -idempotent-separating congruence of a $*$ -simple type A I -semigroup is characterized in the theorem.

Theorem 3.4. Let $S = GBR^*(T, \theta)$ where $T = \cup_{i=0}^{d-1} M_i$. The relation ρ on $S = GBR^*(T, \theta)$ defined by the rule:

$$(x_i, m, n) \rho (y_j, p, q) \text{ if and only if } m = p, n = q, i = j \text{ and } (x_i, y_j) \in \ker \theta$$

is a $*$ -locally idempotent separating congruence

Proof. It can be easily shown that ρ is reflexive and symmetric. To show transitivity, we let $(x_i, m, n) \rho (y_j, p, q), (y_j, p, q) \rho (z_k, u, v)$ for all $(x_i, m, n), (y_j, p, q), (z_k, u, v) \in S$. Then $m = p, n = q, i = j, (x_i, y_j) \in \ker \theta$ and $p = u, q = v, j = k, (y_j, z_k) \in \ker \theta$.

Consequently, $m = u, n = v, i = k$. Hence $(x_i, z_k) \in \ker \theta$, which means that ρ is transitive.

Next is to show that ρ is a congruence. Now let $a = (x_i, m, n), b = (y_j, p, q)$. That ρ is a congruence entails showing that

$$a \rho b \text{ implies } ag \rho bg \quad (\text{for right congruence})$$

$$a \rho b \text{ implies } ga \rho gb \quad (\text{for left congruence})$$

$$\forall g = (z_k, w, l) \in S = GBR^*(T, \theta).$$

Consequently,

$$\begin{aligned} ag &= (x_i, m, n)(z_k, w, l) \\ &= \begin{cases} (x_i \cdot f_{n-w,w}^{-1} \cdot z_k \theta^{n-w} \cdot f_{n-w,l}, m, n + l - w) & \text{if } n \geq w \\ (f_{w-n,m}^{-1} \cdot x_i \theta^{w-n} \cdot f_{w-n,n} \cdot z_k, m + w - n, l) & \text{if } n \leq w \end{cases} \\ bg &= (y_j, p, q)(z_k, w, l) \\ &= \begin{cases} (y_j \cdot f_{q-w,w}^{-1} \cdot z_k \theta^{q-w} \cdot f_{q-w,l}, p, q + l - w) & \text{if } q \geq w \\ (f_{w-q,p}^{-1} \cdot y_j \theta^{w-q} \cdot f_{w-q,q} \cdot z_k, p + w - q, l) & \text{if } q \leq w \end{cases} \end{aligned}$$

So, if $(x_i, m, n) \rho (y_j, p, q)$, then

$$(x_i, m, n)(z_k, w, l) \rho (y_j, p, q)(z_k, w, l) =$$

$$\rho \begin{cases} (x_i \cdot f_{n-w,w}^{-1} \cdot z_k \theta^{n-w} \cdot f_{n-w,l}, m, n+l-w) & \text{if } n \geq w \\ (f_{w-n,m}^{-1} \cdot x_i \theta^{w-n} \cdot f_{w-n,n} \cdot z_k, m+w-n, l) & \text{if } n \leq w \\ (y_j \cdot f_{q-w,w}^{-1} \cdot z_k \theta^{q-w} \cdot f_{q-w,l}, p, q+l-w) & \text{if } q \geq w \\ (f_{w-q,p}^{-1} \cdot y_j \theta^{w-q} \cdot f_{w-q,q} \cdot z_k, p+w-q, l) & \text{if } q \leq w \end{cases}$$

But $(x_i, m, n) \rho (y_j, p, q)$ if and only if $m = p, n = q, i = j$ and $(x_i, y_j) \in \ker \theta$.

Thus,

$$\rho \begin{cases} (x_i \cdot f_{n-w,w}^{-1} \cdot z_k \theta^{n-w} \cdot f_{n-w,l}, m, n+l-w) & \text{if } n \geq w \\ (f_{w-n,m}^{-1} \cdot x_i \theta^{w-n} \cdot f_{w-n,n} \cdot z_k, m+w-n, l) & \text{if } n \leq w \\ (y_j \cdot f_{n-w,w}^{-1} \cdot z_k \theta^{n-w} \cdot f_{n-w,l}, m, n+l-w) & \text{if } n \geq w \\ (f_{w-n,m}^{-1} \cdot y_j \theta^{w-n} \cdot f_{w-n,n} \cdot z_k, m+w-n, l) & \text{if } n \leq w \end{cases}$$

Hence ρ is a right congruence.

That ρ is a left congruence follows similarly. Thus ρ is a congruence.

Furthermore, $(e_i, m, m) \rho (e_i, n, n)$ implies $m = n$ which implies $(e_i, m, m) = (e_i, n, n)$. Thus any two distinct idempotent elements which are \mathcal{D}^* -related cannot lie in the same ρ -class. Hence the proof.

We will now construct the strictly $*$ -locally idempotent-separating congruences on $*$ -simple type A I -semigroups.

3.5. Notation. Let $k_0, k_1, k_2, k_3, \dots, k_t$ be a sequence of non-empty integers, satisfying $0 \leq k_0 < k_1 \dots < k_t < d - 1, k_0 = -1, k_{t+1} = d - 1$.

Define a relation $\rho = \rho(k_0, k_1, \dots, k_t)$ on $S = GBR^*(T, \theta)$ by

$$(x_i, m, n) \rho (y_j, p, q) \text{ implies } \begin{cases} m = p, n = q \text{ for } k_{v-1} < i, j \leq k_v, 0 \leq v \leq t + 1 \\ \text{or } m = p + 1, n = q + 1 \text{ for } i \leq k_0 \text{ and } j > k_t \\ \text{or } m + 1 = p, n + 1 = q \text{ for } j \leq k_0 \text{ and } i > k_t \end{cases}$$

Lemma 3.6. With the notation introduced, $\rho = \rho(k_0, k_1, \dots, k_t)$ is a strictly $*$ -locally idempotent-separating congruence on $S = GBR^*(T, \theta)$.

Proof. Suppose ρ is a strictly $*$ -locally idempotent-separating congruence on $S = GBR^*(T, \theta)$. Then we have that

$$(x_i, m, n) \rho (y_j, p, q) \text{ implies } (y_j, n, m) \rho (x_i, q, p)$$

since it is evident that the relation ρ defined above is a congruence on a type A semigroup (where (y_j, n, m) is inverse of (x_i, m, n) and (x_i, q, p) is the inverse of (y_j, p, q)).

Now we have that $(x_i, m, n)^\dagger \rho (y_j, p, q)^\dagger$ and $(x_i, m, n)^* \rho (y_j, p, q)^*$ implies $(e_i, m, m) \rho (e_i, p, p)$ and $(e_i, n, n) \rho (e_i, q, q)$.

That is, we have that $e_{md+i} \rho e_{pd+j}$ and $e_{nd+i} \rho e_{qd+j}$.

Suppose $md + i \geq (p + 1)d + j$ then $e_{pd+j} \rho e_{(p+1)d+j}$ then $i < j, m \leq p + 1, i > j, m \leq p$.

Similarly, we have $j < i, p \leq m + 1. j > i, p \leq m.$

Consequently, we have $i < j, m \leq p + 1 \leq m + 1.$ That is $m = p$ or $p + 1.$

$i > j, m \leq p \leq m + 1.$ That is $p = m$ or $m + 1.$

Interchanging the roles of m and n, p and q we have that

$i < j, n = q$ or $q + 1. i > j, q = n$ or $n + 1.$

Now using Lemma 3.2 and considering some cases, we have the desired result.

We now consider cancellative monoid congruences. These can be characterized as follows:

Theorem 3.7. Let $S = GBR^*(T, \theta)$ be a $*$ -simple type A l -semigroup. Define a relation σ on S by

$$(x_i, m, n) \sigma (y_j, p, q)$$

if and only if $m - n = p - q$ and $x_i = y_j.$ Then

i) σ is the minimum congruence on $S.$

ii) S/σ is a cancellative monoid.

Proof. i) That σ is reflexive and symmetric can be easily checked. To show transitivity, let $(x_i, m, n) \sigma (y_j, p, q)$ and $(y_j, p, q) \sigma (z_k, r, c)$ for $(x_i, m, n), (y_j, p, q), (z_k, r, c) \in S.$ Then we have $m - n = p - q, x_i = y_j$ and $p - q = r - c, y_j = z_k.$ This implies $m - n = r - c$ and $x_i = z_k.$ Thus σ is transitive.

Now let $a = (x_i, m, n), b = (y_j, p, q).$ That σ is a congruence entails showing that

$$a \sigma b \Rightarrow au \sigma bu \quad (\text{for right congruence})$$

$$a \sigma b \Rightarrow ua \sigma ub \quad (\text{for left congruence})$$

$\forall u = (z_k, r, c) \in S.$ So, we have that

$$au = (x_i, m, n)(z_k, r, c) = \begin{cases} (x_i \cdot f_{n-r, r}^{-1} \cdot z_k \theta^{n-r} \cdot f_{n-r, c}, m, n + c - r) & \text{if } n \geq r \\ (f_{r-n, n}^{-1} \cdot x_i \theta^{r-n} \cdot f_{r-n, n} \cdot z_k, m + r - n, c) & \text{if } n \leq r \end{cases}$$

$$bu = (y_j, p, q)(z_k, r, c) = \begin{cases} (y_j \cdot f_{q-r, r}^{-1} \cdot z_k \theta^{q-r} \cdot f_{q-r, c}, p, q + c - r) & \text{if } q \geq r \\ (f_{r-q, q}^{-1} \cdot y_j \theta^{r-q} \cdot f_{r-q, q} \cdot z_k, p + r - q, c) & \text{if } q \leq r \end{cases}$$

Suppose $(x_i, m, n) \sigma (y_j, p, q),$ we have

$$m - (n + c - r) = (m - n) + (r - c) \quad \text{and} \quad p - (q + c - r) = (p - q) + (r - c)$$

$$m + r - n - c = (m - n) + (r - c) \quad \text{and} \quad p + r - q - c = (p - q) + (r - c).$$

Since $m - n = p - q,$ we have that $(m - n) + (r - c) = (p - q) + (r - c).$

Consequently, σ is a right congruence. That σ is a left congruence follows similarly. Thus σ is a congruence.

Suppose ρ is any other congruence. Then we have $(1, m, m) \rho (1, 0, 0)$ for all $m \in I$. If $(x_i, m, n) \sigma (y_j, p, q)$, then $(x_i, m, n)(1, p, p) = (y_j, p, q)(1, p, p)$ for some $p \in I$

Since $(1, m, m) \rho (1, 0, 0)$, then $(x_i, m, n)(1, p, p) \rho (x_i, m, n)$.

Similarly, $(y_j, p, q)(1, p, p) \rho (y_j, p, q)$ so that $(x_i, m, n) \rho (y_j, p, q)$. Hence $\sigma \subseteq \rho$.

ii) Obviously the class of σ containing the idempotents is the identity element for S/σ . So we have $(1, m, n)\sigma (y_j, p, q)\sigma = (y_j, p, q)\sigma$. Thus S/σ is a monoid.

To show that S/σ is cancellative, let $a = (x_i, m, n), b = (y_j, p, q)$. That S/σ is cancellative entails showing that

$$a\sigma u\sigma = b\sigma u\sigma \implies a\sigma = b\sigma \quad (\text{for right cancellative})$$

$$u\sigma a\sigma = u\sigma b\sigma \implies a\sigma = b\sigma \quad (\text{for left cancellative})$$

$\forall u = (z_k, r, c) \in S$. So, we have that

$$\begin{aligned} a\sigma u\sigma &= (x_i, m, n)\sigma (z_k, r, c)\sigma = (y_j, p, q)\sigma (z_k, r, c)\sigma \\ &= b\sigma u\sigma. \end{aligned}$$

The rest of the proof follows from a routine calculation.

For the remainder of this section the group of integers under addition will be denoted by \mathbb{Z} .

We now describe the nature of S/σ in the case where θ is the identity mapping.

Theorem 3.8. Let $S = GBR^*(T, \theta)$ be a $*$ -simple type A l-semigroup in which θ is the identity mapping. Define a multiplication on the set $T \times \mathbb{Z}$ by the rule that

$$(x_i, md + i)(y_j, nd + i) = (x_i y_j, (md + i) + (nd + i))$$

for $x_i, y_j \in T, m, n \in \mathbb{Z}$. Then $S/\sigma \cong T \times \mathbb{Z}$.

Proof. Define a map $\varphi : S \rightarrow T \times \mathbb{Z}$ by the rule that $(x_i, m, n)\varphi = (x_i y_j, (md + i) - (nd + i))$.

Evidently, φ is well defined. It is known that $T \times \mathbb{Z}$ is a cancellative monoid with identity $(1, 0)$.

Now let (x_i, m, n) and (y_j, p, q) be any two elements of S . Then

$$\begin{aligned} ((x_i, m, n)(y_j, p, q))\varphi &= \begin{cases} (x_i \cdot f_{n-p,p}^{-1} \cdot y_j \theta^{n-p} \cdot f_{n-p,q}, m, n + q - p) & \text{if } n \geq p \\ (f_{p-n,n}^{-1} \cdot x_i \theta^{p-n} \cdot f_{p-n,n} \cdot y_j, m + p - n, q) & \text{if } n \leq p \end{cases} \times \varphi \\ &= \begin{cases} (x_i \cdot f_{n-p,p}^{-1} \cdot y_j \theta^{n-p} \cdot f_{n-p,q}) \varphi & \text{if } n \geq p \\ (f_{p-n,n}^{-1} \cdot x_i \theta^{p-n} \cdot f_{p-n,n} \cdot y_j) \varphi & \text{if } n \leq p \end{cases} \\ &= \begin{cases} (x_i y_j, md + i - (n + q - p)d + i) & \text{if } n \geq p \\ (x_i y_j, (m + p - n)d + j - qd + j) & \text{if } n \leq p \end{cases} \end{aligned}$$

$$\begin{aligned}
&= \begin{cases} (x_i y_j, (m-n)d+i+(p-q)d+i) & \text{if } n \geq p \\ (x_i y_j, (m-n)d+j+(p-q)d+j) & \text{if } n \leq p \end{cases} \\
&= (x_i y_j, (m-n)d+i+(p-q)d+j) \\
&= (x_i, (m-n)d+i)(y_j, (p-q)d+j) \\
&= (x_i, (md+i)-(nd+i))(y_j, (pd+j)-(qd+j)) \\
&= (x_i, m, n)\varphi (y_j, p, q)\varphi .
\end{aligned}$$

Thus φ is a homomorphism.

Furthermore,

$$(x_i, m, n)\varphi = (y_j, p, q)\varphi$$

if and only if $(x_i, (md+i)-(nd+i)) = (y_j, (pd+j)-(qd+j))$

if and only if $(md+i)-(nd+i) = (pd+j)-(qd+j)$ and $x_i = y_j$

if and only if $(x_i, m, n)\sigma = (y_j, p, q)\sigma$.

That is $\varphi \circ \varphi^{-1} = \sigma$.

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