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Symmetry problem 1

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Abstract

A symmetry problem is solved. A new method is used. The idea of this method is to reduce to a contradiction the PDE and the over-determined boundary data on the boundary.

The new method allows one to solve other symmetry problems.

1 Introduction

Symmetry problems for PDE were studied in many publications by many authors, see, for example, [1]. In this paper a new method is given for a study of symmetry problems for PDE. Throughout we assume that D is a bounded connected C^2-smooth domain in $\mathbb{R}^3$, S is the boundary of D, N is the unit normal to S, pointing out of D, u_N is the normal derivative of u on S, $D' = \mathbb{R}^3 \setminus D$, S^2 is the unit sphere in $\mathbb{R}^3$, $J_n(r)$ is the Bessel function regular at $r = 0$, $j_\ell(r)$ is the spherical Bessel function, $j'_\ell(kr) = \frac{dj_\ell(kr)}{dr}$, $k > 0$ is a constant, $\beta \cdot y = (\beta, y)$ is the dot product.

In [2]–[10] the author studied various symmetry problems.

Let us formulate the symmetry problem studied in this paper. Our main result is formulated in Theorem 1.

Theorem 1. *Assume that*

$$\Delta u + k^2 u = 0 \quad in \quad D, \quad u|_S = 1, \quad u_N = 0. \tag{1}$$

Then S is a sphere of radius a where a solves the equation $j'_0(ka) = 0$.

In [5], [7] it was shown that the Pompeiu problem is equivalent to the problem (1).

In Section 2 proofs are given.

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2 Proofs

Proof of Theorem 1. Let $g(x,y,k) := \frac{e^{ik|x-y|}}{4\pi|x-y|}$. If problem (1) has a solution then this solution is unique by the uniqueness of the solution to the Cauchy problem for elliptic equation (1). The solution to equation (1) by Green's formula is:

$$u(x) = -\int_S g_N(x,t)dt, \quad x \in D; \quad u(x) = -\int_S g_N(x,t)dt = 0, \quad x \in D'. \tag{2}$$

Let $B_R = \{x : |x| \leq R\}$, $D \subset B_R$. If D is a ball B_a of radius a and $j_0'(ka) = 0$, then problem (1) in B_a has a solution:

$$u = \frac{j_0(kr)}{j_0(ka)}, \quad r = |x|. \tag{3}$$

In what follows we assume that $D \subset \mathbb{R}^2$ and S is a closed smooth curve. Let $\mathbf{r}(\mathbf{s}) = x(s)e_1 + y(s)e_2$ be a parametric representation of S, s be the arc length along S and also the corresponding to the arc length s point on S, $\{e_1, e_2\}$ is a Cartesian basis in $\mathbb{R}^2$. The first boundary condition in (1) is $u(x(s), y(s)) = 1$. Differentiating with respect to s one gets $u_x\dot{x} + u_y\dot{y} = 0$ and another differentiation yields

$$u_{xx}\dot{x}^2 + 2u_{xy}\dot{x}\dot{y} + u_{yy}\dot{y}^2 = 0, \quad \dot{x} = \frac{dx}{ds}. \tag{4}$$

Here we used the formula $u_x\ddot{x} + u_y\ddot{y} = 0$. This formula can be derived as follows: $\nabla u \cdot \ddot{\mathbf{r}} = \nabla u \cdot \kappa\nu$, where $\kappa = \kappa(s) > 0$ is the curvature of S, $\nu = -N$ is the unit normal pointing into D, $\nabla u \cdot \nu = -u_N = 0$ on S. From (1) it follows that

$$u_{xx} + u_{yy} = -k^2 \quad on \quad S. \tag{5}$$

Let us prove that (4) and (5) are not compatible at some points, except when S is a circle of radius a, where a solves the equation $j_0'(ka) = 0$.

Denote $u_{xx} = p = p(s)$, $u_{xy} = q = q(s)$. Then (5) implies $u_{yy} = -k^2 - p$ on S. Let A be a 2×2 matrix with elements $A_{11} = p$, $A_{12} = A_{21} = q$, $A_{22} = -k^2 - p$. The equation for finding the eigenvalues $\lambda_{1,2}$ of A is:

$$\lambda^2 + k^2\lambda - p^2 - q^2 - k^2p = 0. \tag{6}$$

The eigenvalues λ_1 and λ_2 are:

$$\lambda_{1,2} = -\frac{k^2}{2} \pm (k^4/4 + p^2 + q^2 + k^2p)^{1/2}. \tag{7}$$

Clearly, $\lambda_1 + \lambda_2 = -k^2$, $\lambda_1\lambda_2 = -p^2 - q^2 - k^2p$, $k^4/4 + p^2 + q^2 + k^2p = (\frac{k^2}{2} + p)^2 + q^2 \geq 0$. Thus, $\lambda_2 < 0$.

The corresponding eigenvectors (non-normalized but orthogonal) can be calculated explicitly. One has

$$e_1 = \{1, \gamma\}, \quad \gamma := \frac{q}{k^2 + p + \lambda_1} = \frac{\lambda_1 - p}{q}. \tag{8}$$

If $q \neq 0$, then

$$e_2 = \{\frac{k^2 + p + \lambda_2}{q}, 1\} = \{-\gamma, 1\}. \tag{9}$$

If $q \neq 0$ then one checks that $\frac{k^2+p+\lambda_2}{q} = \frac{q}{\lambda_2 - p}$ and $\frac{q}{\lambda_2 - p} + \frac{q}{k^2+p+\lambda_1} = 0$, so $\gamma = -\frac{k^2+p+\lambda_2}{q}$.

If $q = 0$ then $\lambda_1 = p$, $\lambda_2 = -p - k^2$, $e_1 = \{1, 0\}$, $e_2 = \{0, 1\}$, and equation (13) (see below) leads also to a contradiction as in the case $q \neq 0$.

Clearly, $e_1 \cdot e_2 = 0$, $\|e_1\|^2 = \|e_2\|^2 = 1 + \gamma^2$, so γ^2 is invariant under rotations of the Cartesian coordinate system.

Denote $\{\dot{x}, \dot{y}\} := w$. Note that $\dot{x}^2 + \dot{y}^2 = 1$. Let c_1, c_2 be scalar coefficients. Then

$$c_1 e_1 + c_2 e_2 = w, \quad w := \{\dot{x}, \dot{y}\}. \tag{10}$$

Solving explicitly this algebraic system for c_1 and c_2 one gets:

$$c_1 = (\dot{x} + \gamma \dot{y})\, \Delta^{-1}, \quad \Delta = 1 + \gamma^2, \tag{11}$$

and

$$c_2 = (\dot{y} - \gamma \dot{x})\, \Delta^{-1}. \tag{12}$$

Equation (4) can be written as $(Aw, w) = 0$. Substitute w from (10) into the equation $(Aw, w) = 0$ and use the orthogonality of e_1 and e_2 to get

$$(\dot{y} - \gamma \dot{x})^2\, \lambda_2 + (\dot{x} + \gamma \dot{y})^2\, \lambda_1 = 0. \tag{13}$$

We now prove that (13) leads to a contradiction unless S is a circle of radius a where a solves the equation $J_0'(ka) = 0$ if $D \subset \mathbb{R}^2$ and a solves the equation $j_0'(ka) = 0$ if $D \subset \mathbb{R}^3$.

Choose Cartesian coordinates in which $\dot{x}(s) = -\gamma \dot{y}$. Such coordinate system does exist because the only restriction on $\dot{x}$ and $\dot{y}$ is $\dot{x}^2 + \dot{y}^2 = 1$ at all $s \in S$. Then, since $\lambda_2 < 0$ equation (13) with $\dot{x}(s) = -\gamma \dot{y}$ implies $\dot{y}(1 + \gamma^2) = 0$. Thus, $\dot{y} = 0$. Therefore, $\dot{x} = \dot{y} = 0$. This contradicts the relation $\dot{x}^2 + \dot{y}^2 = 1$. This contradiction holds for any smooth S except for a circle of a special radius, see (3).

Theorem 1 is proved. □

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