Argument estimates of certain classes of P-Valent meromorphic functions involving certain operator

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ABSTRACT

In this paper, by making use of subordination, we investigate some inclusion relations and argument properties of certain classes of p-valent meromorphic functions involving certain operator.

Indexing terms/Keywords

Argument estimates, Hadamard product, certain operator, meromorphic functions.

SUBJECT CLASSIFICATION

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1. Introduction

For any integer \( m > -p \), let \( \sum_{m} \) denote the class of meromorphic functions \( f(z) \) of the form:

\[
f(z) = z^{-p} + \sum_{k=m}^{\infty} a_k z^k \quad (m > -p; \quad p \in \mathbb{N} = \{1, 2, \cdots \}),
\]

which are analytic and \( P \)-valent in the punctured open unit disc \( U^* = \{ z \in \mathbb{C} : 0 < |z| < 1 \} = \mathbb{U} \setminus \{0\} \). For convenience, we write \( \sum_{p, m} = \sum_{m} \) and \( \sum_{0} = \sum \). If \( f(z) \) and \( g(z) \) are analytic in \( U \), we say that \( f(z) \) is subordinate to \( g(z) \), written \( f \prec g \) or \( f(z) \prec g(z) \) \( (z \in U) \), if there exists a Schwarz function \( w(z) \) in \( U \) with \( w(0) = 0 \) and \( |w(z)| < 1 \) \( (z \in U) \), such that \( f(z) = g(w(z)) \) \((z \in U)\). Furthermore, if \( g(z) \) is univalent in \( U \), then the following equivalence relationship holds true (see [7] and [18]):

\[
f(z) \prec g(z) \iff f(0) = g(0) \quad \text{and} \quad f(U) \subset g(U).
\]

For functions \( f(z) \in \sum_{m} \), given by (1.1) and \( g(z) \in \sum_{m} \) defined by

\[
g(z) = z^{-p} + \sum_{k=m}^{\infty} b_k z^k \quad (m > -p; \quad p \in \mathbb{N}),
\]

the Hadamard product (or convolution) of \( f(z) \) and \( g(z) \) is given by
(f * g)(z) = z^{-p} + \sum_{k=m}^{\infty} a_k b_k z^k = (g * f)(z).

For complex parameters \( \alpha_1, \alpha_2, \ldots, \alpha_q \) and
\( \beta_1, \beta_2, \ldots, \beta_s \) \((\alpha_j, \beta_j \notin \mathbb{Z}_0 = \{0, \ldots, -2, \ldots\}, i = 1, 2, \ldots, q; \ j = 1, 2, \ldots, s)\), the generalized hypergeometric function \( _q F_s (\alpha_1, \alpha_2, \ldots, \alpha_q; \beta_1, \beta_2, \ldots, \beta_s; z) \) is defined by (see [22])
\[
_q F_s (\alpha_1, \alpha_2, \ldots, \alpha_q; \beta_1, \beta_2, \ldots, \beta_s; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_q)_k}{(\beta_1)_k \cdots (\beta_s)_k (1)_k} z^k
\]
\((q \leq s + 1); \ s, q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; \ z \in \mathbb{U})\),
where \((\theta)_k\) is the Pochhammer symbol defined in terms of the Gamma function \( \Gamma \), by
\[
(\theta)_k = \frac{\Gamma(\theta + k)}{\Gamma(\theta)} \quad (\theta \in \mathbb{C^*} = \mathbb{C} \setminus \{0\})
\]
\((v \in \mathbb{N}; \ \theta \in \mathbb{C})\).

Liu and Srivastava [16] and Aouf [4] investigated recently the operator \( Y_{p,q,s} (\alpha_1, \alpha_2, \ldots, \alpha_q) ; \beta_1, \beta_2, \ldots, \beta_s) : \sum_{p,m} \to \sum_{p,m} \), defined as follows:
\[
Y_{p,q,s} (\alpha_1) = Y_{p,q,s} (\alpha_1, \alpha_2, \ldots, \alpha_q; \beta_1, \beta_2, \ldots, \beta_s; z) = z^{-p} \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_q)_k}{(\beta_1)^{k+p} \cdots (\beta_s)^{k+p} (1)_k} a_k z^k.
\]
\[(1.2)\]
With aid of the function \( Y_{p,q,s} (\alpha_1) \) given by (1.2), consider the function \( Y^*_{p,q,s} (\alpha_1) \) defined by:
\[
Y_{p,q,s} (\alpha_1) \ast Y^*_{p,q,s} (\alpha_1) = \frac{1}{z^p (1-z)^{\lambda + p}} (\lambda > -p; p \in \mathbb{N}; z \in \mathbb{U}^*).
\]
\[(1.3)\]
This function leads us to the following family of linear operators \( M_{p,q,s}^{\lambda,m} (\alpha_1) : \sum_{p,m} \to \sum_{p,m} \), which are given by:
\[
M_{p,q,s}^{\lambda,m} (\alpha_1) = Y^*_{p,q,s} (\alpha_1) \ast f(z) (f \in \sum_{p,m}).
\]
\[(1.4)\]
The linear operator \( M_{p,q,s}^{\lambda,m} (\alpha_1) \) was defined by Patel and Patil [20] and Mostafa [17]. If \( f(z) \) is given by (1.1), then from (1.4), we deduce that
\[
M_{p,q,s}^{\lambda,m} (\alpha_1) f(z) = z^{-p} \sum_{k=m}^{\infty} \frac{(\beta_1)^{k+p} \cdots (\beta_s)^{k+p} (\lambda + p)_k}{(\alpha_1)^{k+p} \cdots (\alpha_q)^{k+p}} a_k z^k.
\]
\[ (f \in \sum_{p,m} ; \lambda, m > -p; p \in \mathbb{N}; z \in U^*). \]  

(1.5)

It is easily verified from (1.5) that [see [20] and [17]]

\[ z\left( M_{\alpha_1}^{\lambda,m} (\alpha_1) f(z) \right) = (\lambda + p) M_{\alpha_1}^{\lambda,m} (\alpha_1) f(z) - (\lambda + 2p) M_{\alpha_1}^{\lambda,m} (\alpha_1) f(z) \]

and

\[ z\left( M_{\alpha_1}^{\lambda,m} (\alpha_1 + 1) f(z) \right) = \alpha M_{\alpha_1}^{\lambda,m} (\alpha_1) f(z) - (\alpha_1 + p) M_{\alpha_1}^{\lambda,m} (\alpha_1 + 1) f(z). \]  

(1.6)

(1.7)

For a function \( f \in \sum_p \) and \( \mu > 0 \), let \( F_{\mu,p} : \sum_p \rightarrow \sum_p \) be the integral operator defined by [see [13]]:

\[ F_{\mu,p}(f)(z) = \frac{\mu}{z} \int_0^z t^{\mu+p-1} f(t) dt = z^{-p} + \sum_{k=1}^p \frac{\mu}{\mu+k+p} a_k z^k \]  

(1.8)

\( (f \in \sum_p ; \mu > 0; p \in \mathbb{N}; z \in U^*). \)

It follows from (1.8) that:

\[ z\left( M_{\alpha_1}^{\lambda,m} (\alpha_1) F_{\mu,p}(f)(z) \right)' = \mu M_{\alpha_1}^{\lambda,m} (\alpha_1) f(z) - (\mu + p) M_{\alpha_1}^{\lambda,m} (\alpha_1) F_{\mu,p}(f)(z). \]  

(1.9)

We note that:

Putting \( \lambda = 1 - p \) \( (p \in \mathbb{N}) \) in (1.5), then the operator \( M_{\alpha_1}^{1-p,m} (\alpha_1) \) reduces to the operator \( M_{\alpha_1}^m (\alpha_1) \), defined by:

\[ M_{\alpha_1}^m (\alpha_1) f(z) = z^{-p} + \sum_{k=m}^\infty (\alpha_1)_{k+p} a_k z^k \quad (f \in \sum_{p,m} ; m > -p; p \in \mathbb{N}; z \in U^*) \]

Also, by specializing the parameters \( \lambda, m, p, \alpha_1 \) \( (i = 1, 2, \ldots, q) \), \( \beta_j \) \( (j = 1, 2, \ldots, s) \), \( q \) and \( s \), we have:

(i) \( M_{p,q,s}^{0,0}(p, p; p) f(z) = M_{p,q,s}^{0,0}(p+1, p; p) f(z) = f(z) (p \in \mathbb{N}) \);

(ii) \( M_{p,q,s}^{1,1}(p, p; p) f(z) = \frac{zf(z) + f'(z)}{p} (p \in \mathbb{N}) \);

(iii) \( M_{p,q,s}^{2,2}(p+1, p; p) f(z) = \frac{(2p+1)f(z) + f'(z)}{p+1} (p \in \mathbb{N}) \);

(iv) \( M_{p,q,s}^{n,0}(a, 1; a) f(z) = D_n f(z) \) \( (n > -p, a > 0, p \in \mathbb{N}) \) (see Yang [23] and Aouf ([2] and [3]), which for \( p = 1 \) reduces to the operator \( D_n f(z) \) \( (n > -1) \) (see Cho [8]);

(v) \( M_{p,q,s}^{0,0}(p+1, p; p) f(z) = \frac{1}{zp} \int_0^z t^{2p-1} f(t) dt (p \in \mathbb{N}) \);

(vi) \( M_{p,q,s}^{1,1}(\mu+1, \mu; \mu) f(z) = F_{\mu,p}(f)(z) \) \( (p \in \mathbb{N}, \mu > 0) \), this integral operator is defined by (1.8);

(vii) \( M_{p,q,s}^{m,m}(c, \mu; \mu) f(z) = L_p \left( a; c \right) f(z) \left( p \in \mathbb{N}, a \in \mathbb{R}, c \in \mathbb{R} \setminus Z_0 \right) \) (see Liu [15]);
In order to prove our main results, we need to the following lemmas.

**Lemma 1** [12]. Let \( \beta \) and \( \nu \) be complex constants and let \( h(z) \) be convex (univalent) in \( U \) with \( h(0) = 1 \) and \( \text{Re}(\beta h(z) + \nu) > 0 \). If \( q(z) = 1 + q_1 z + \ldots \) is analytic in \( U \), then

\[
q(z) + \frac{zq'(z)}{\beta q(z) + \nu} < h(z),
\]

where \( q(z) = 1 + q_1 z + \ldots \).
implies
\[ q(z) < h(z). \]

**Lemma 2** [18]. Let \( h(z) \) be convex (univalent) in \( U \) and \( \psi(z) \) be analytic in \( U \) with \( \Re\{\psi(z)\} \geq 0 \). If \( q \) is analytic in \( U \) and \( q(0) = h(0) \), then
\[ q(z) + \psi(z)q'(z) < h(z), \]
implies
\[ q(z) < h(z). \]

**Lemma 3** [19]. Let \( q(z) \) be analytic in \( U \), with \( q(0) = 1 \) and \( q(z) \neq 0 \) (\( z \in U \)). If there exists a point \( z_0 \in U \), such that
\[ |\arg q(z)| < \frac{\pi}{2} \text{ for } |z| < |z_0| \]  
(2.1)
and
\[ |\arg q(z_0)| = \frac{\pi}{2} \text{ (} 0 < \tau \leq 1 \text{).} \]  
(2.2)
Then we have
\[ \frac{z_0q'(z_0)}{q(z_0)} = ix\tau, \]  
(2.3)
where
\[ x \geq \frac{1}{2}(b + \frac{1}{b}) \text{ when } \arg q(z_0) = \frac{\pi}{2}, \]  
(2.4)
\[ x \geq -\frac{1}{2}(b + \frac{1}{b}) \text{ when } \arg q(z_0) = -\frac{\pi}{2} \]  
(2.5)
and
\[ q(z_0)^{\frac{1}{2}} = \pm ib \text{ (} b > 0 \text{).} \]  
(2.6)

3. Some inclusion relationships
By using Lemma 1, we obtain the following results:
Theorem 1. Let $h(z) \in M$ with $\max_{z \in \mathcal{U}} \Re \{h(z)\} < \min\{\frac{p+2p}{p}, \frac{\alpha + p}{p}\}$ ( $\mathcal{P} \neq \emptyset$, $p \in \mathbb{N}$). Then

$$\sum_{p, q, s}^{\lambda + 1, m} (\alpha_i; h) \subset \sum_{p, q, s}^{\lambda, m} (\alpha_i; h) \subset \sum_{p, q, s}^{\lambda, m} (\alpha_i + 1; h).$$

Proof. To prove the first part, we show that $\sum_{p, q, s}^{\lambda + 1, m} (\alpha_i; h) \subset \sum_{p, q, s}^{\lambda, m} (\alpha_i; h)$. Let $f \in \sum_{p, q, s}^{\lambda + 1, m} (\alpha_i; h)$ and set

$$R(z) = \frac{z\left[\mathcal{M}_{p, q, s}^{\lambda, m}(\alpha_i) f(z)\right]}{p \mathcal{M}_{p, q, s}^{\lambda, m}(\alpha_i) f(z)} (z \in \mathcal{U}), \quad (3.1)$$

where $R(z)$ is analytic with $R(0) = 1$. Using (1.6) in (3.1), we obtain

$$pR(z) - (\lambda + 2 p) = -(\lambda + p) \frac{\mathcal{M}_{p, q, s}^{\lambda + 1, m}(\alpha_i) f(z)}{\mathcal{M}_{p, q, s}^{\lambda, m}(\alpha_i) f(z)}. \quad (3.2)$$

Differentiating (3.2) logarithmically with respect to $z$ and multiplying by $z$, we have

$$R(z) + \frac{zR'(z)}{pR(z) + \lambda + 2 p} = -\frac{z\left[\mathcal{M}_{p, q, s}^{\lambda + 1, m}(\alpha_i) f(z)\right]}{p \mathcal{M}_{p, q, s}^{\lambda + 1, m}(\alpha_i) f(z)} \subset h(z), \quad (3.3)$$

from Lemma 1, it follows that $R(z) \subset h(z)$ in $\mathcal{U}$, that is, that $f \in \sum_{p, q, s}^{\lambda, m} (\alpha_i; h)$.

To prove the second part, let $f \in \sum_{p, q, s}^{\lambda, m} (\alpha_i; h)$ and put

$$s(z) = -\frac{z\left[\mathcal{M}_{p, q, s}^{\lambda, m}(\alpha_i + 1) f(z)\right]}{p \mathcal{M}_{p, q, s}^{\lambda, m}(\alpha_i + 1) f(z)} (z \in \mathcal{U}), \quad (z \in \mathcal{U}),$$

then, by using the arguments similar to those detailed above and using (1.7) instead of (1.6), it follows that

$s(z) \subset h(z)$ in $\mathcal{U}$, which implies $f \in \sum_{p, q, s}^{\lambda, m} (\alpha_i + 1; h)$. Therefore we complete the proof of Theorem 1.

Taking $h(z) = \frac{1 + A}{1 + B}$ ($-1 < B < A \leq 1$) in Theorem 1, we have

Corollary 1. Let $\frac{1 + A}{1 + B} < \min\{\frac{\lambda + 2 p}{p}, \frac{\alpha + p}{p}\}$ and $-1 < B < A \leq 1$. Then

$$\sum_{p, q, s}^{\lambda + 1, m} (\alpha_i; A, B) \subset \sum_{p, q, s}^{\lambda, m} (\alpha_i; A, B) \subset \sum_{p, q, s}^{\lambda, m} (\alpha_i + 1; A, B).$$

Theorem 2. Let $h(z) \in M$ with $\Re \{h(z)\} < \frac{\mu p}{p} (\mu > 0)$, if $f \in \sum_{p, q, s}^{\lambda, m} (\alpha_i; h)$, then

$F_{\mu, p} (f) \in \sum_{p, q, s}^{\lambda, m} (\alpha_i; h)$, where $F_{\mu, p} (f)$ is defined by (1.8).

Proof. Let $f \in \sum_{p, q, s}^{\lambda, m} (\alpha_i; h)$ and set

$$L(z) = -\frac{z\left[\mathcal{M}_{p, q, s}^{\lambda, m}(\alpha_i) F_{\mu, p} (f)(z)\right]}{p \mathcal{M}_{p, q, s}^{\lambda, m}(\alpha_i) F_{\mu, p} (f)(z)} (z \in \mathcal{U}), \quad (3.4)$$

where $L(z)$ is analytic with $L(0) = 1$. Applying (1.9) to (3.4), we get
\[ pL(z) - (\mu + p) = -\mu \frac{M_{j,m}^1(z, \alpha_1) f(z)}{M_{j,m}^1(z, \alpha_1) F_{\mu, p}(f)(z)}. \]  (3.5)

Differentiating (3.5) logarithmically with respect to \( z \) and multiplying by \( z \), we have

\[ L(z) + \frac{zL(z)}{-pL(z) + \mu + p} = -\frac{\left( M_{j,m}^1(z, \alpha_1) f(z) \right)'}{pM_{j,m}^1(z, \alpha_1) f(z)} < h(z). \]

Hence, by virtue of Lemma 1, we conclude that \( L(z) < h(z) \) in \( U \), which implies

\[ F_{\mu, p}(f) \in \sum p, q, s (\alpha_1; h). \] This completes the proof of Theorem 2.

Taking \( h(z) = \frac{1 + \mu}{1 + B} \) \((-1 < B < A \leq 1)\) in Theorem 2, we have

**Corollary 2.** Let \( \frac{1 + \mu}{1 + B} < \frac{\mu + p}{p} (\mu > 0) \) and \(-1 < B < A \leq 1\), if \( f \in \sum p, q, s (\alpha_1; A, B) \), then

\[ F_{\mu, p}(f) \in \sum p, q, s (\alpha_1; A, B). \]

**4. Some argument properties**

**Theorem 3.** Let \( f(z) \in \sum p, m, 0 < \delta \leq 1, 0 \leq \ell < p \) and

\[ \lambda \geq \frac{p(A-B)}{1+B} - p \quad (-1 < B < A \leq 1; p \in \mathbb{N}). \]

If

\[ \arg \left( -\frac{z \left( M_{j+1,m}^1(z, \alpha_1) f(z) \right)'}{M_{j+1,m}^1(z, \alpha_1) g(z) - \ell} \right) < \frac{\pi}{2} \delta, \]

for some \( g(z) \in \sum j+1, m (\alpha_1; A, B) \), then

\[ \arg \left( -\frac{z \left( M_{j+1,m}^1(z, \alpha_1) f(z) \right)'}{M_{j+1,m}^1(z, \alpha_1) g(z) - \ell} \right) < \frac{\pi}{2} \tau, \]

where \( \tau(0 < \tau \leq 1) \) is the solution of the equation

\[ \delta = \tau + \frac{2}{\pi} \tan^{-1} \left( \frac{\tau \cos \frac{\tau}{2} t(A, B)}{(\lambda - p + 1)(p + 1) - p(A - B)} + \tau \sin \frac{\tau}{2} t(A, B) \right) \] \hspace{1cm} (4.1)

and

\[ t(A, B) = \frac{2}{\pi} \sin^{-1} \left( \frac{p(A - B)}{(\lambda + 2)(1 - B^2) - p(1 - AB)} \right) \] \hspace{1cm} (4.2)

**Proof.** Let
\[
q(z) = \frac{1}{p - \ell} \left( -\frac{z}{\lambda} \left( M_{p,q,s}^{x,m} (\alpha_1) f(z) \right) \right) - \ell \right),
\] (4.3)

where \( q(z) \) is analytic with \( q(0) = 1 \). Applying the identity (1.6), we have

\[
[-(p - \ell)q(z) - \ell] M_{p,q,s}^{x,m} (\alpha_1) g(z) = (\lambda + p) M_{p,q,s}^{x+1,m} (\alpha_1) f(z) - (\lambda + 2p) M_{p,q,s}^{x,m} (\alpha_1) f(z).
\] (4.4)

Differentiating (4.4) with respect to \( z \) and multiplying by \( z \), we obtain

\[
-(p - \ell)zq'(z) M_{p,q,s}^{x,m} (\alpha_1) g(z) + [- (p - \ell)q(z) - \ell] z M_{p,q,s}^{x,m} (\alpha_1) g(z) = (\lambda + p) z M_{p,q,s}^{x+1,m} (\alpha_1) f(z) - (\lambda + 2p) z M_{p,q,s}^{x,m} (\alpha_1) f(z).
\] (4.5)

Then, by using (4.3), (4.4) and (4.5), we have

\[
-\frac{1}{p - \ell} \left( -\frac{z}{\lambda} \left( M_{p,q,s}^{x+1,m} (\alpha_1) f(z) \right) \right) = q(z) + \frac{zq'(z)}{r(z) + \lambda + 2p},
\]

where

\[
r(z) = -\frac{z}{\lambda} \left( M_{p,q,s}^{x+1,m} (\alpha_1) g(z) \right).
\]

From Corollary 1, since \( g(z) \in \sum_{p,q,s}^{x+1,m} (\alpha_1; A,B) \), then \( g(z) \in \sum_{p,q,s}^{x,m} (\alpha_1; A,B) \), which from (1.12) leads to

\[
r(z) < \frac{1 + Az}{1 + Bz}.
\]

Letting

\[-r(z) + \lambda + 2p = \rho e^{i\phi} (z \in U),
\]

then from (1.12) we have

\[
\frac{(\lambda + p)(1 + B) - p(1 - B)}{1 + B} < \rho < \frac{(\lambda + p)(1 - B) + p(1 - B)}{1 - B}
\]

and

\[-t(A,B) < \phi < t(A,B),
\]

where \( t \) is defined by (4.2).
Let \( h \) be a function which maps \( U \) onto the angular domain \( \{ \omega : |\arg \omega| < \frac{\pi}{\delta} \} \) with \( h(0) = 1 \).

Applying Lemma 2, for this \( h \) with \( \psi(z) = \frac{1}{-r(z)+\lambda+2p} \), we see that \( \text{Re}\{\psi(z)\} > 0 \) in \( U \) and hence \( q(z) \neq 0 \) in \( U \). If there exists a point \( z_0 \in U \) such that the conditions (2.1) and (2.2) are satisfied, then by Lemma 3, we have (2.3) under the restrictions (2.4) and (2.5).

At first, suppose that \( q(z_0)^{\frac{1}{2}} = ib \) \( (b > 0) \). Then

\[
\arg\left[ -\frac{1}{p-\ell}\left( \frac{z_0\left( M_{p,q,t}^{l+1,m} (\alpha_1) f(z_0)\right)}{M_{p,q,t}^{l+1,m} (\alpha_1) g(z_0)} + \ell \right) \right] = \arg\left( q(z_0) + \frac{z_0 q'(z_0)}{r(z_0) + \lambda + 2p} \right) = \frac{\pi}{2} \tau + \arg\left( 1 + i\ell \left( e^{i\varphi} \right)^{\frac{1}{2}} \right) = \frac{\pi}{2} + \tan^{-1}\left( \frac{\ell \tau \sin \frac{\pi}{2} (1 - \phi)}{\rho + \ell \tau \cos\frac{\pi}{2} (1 - \phi)} \right) = \frac{\pi}{2} \delta,
\]

where \( \lambda \) and \( \tau(A, B) \) are given by (4.1) and (4.2), respectively. This contradicts to the assumption of the theorem.

Next, suppose that \( q(z_0)^{\frac{1}{2}} = -ib \) \( (b > 0) \). Applying the same method as the above, we have

\[
\arg\left[ -\frac{1}{p-\ell}\left( \frac{z_0\left( M_{p,q,t}^{l+1,m} (\alpha_1) f(z_0)\right)}{M_{p,q,t}^{l+1,m} (\alpha_1) g(z_0)} + \ell \right) \right] \leq -\frac{\pi}{2} \tau + \tan^{-1}\left( \frac{\tau \cos\frac{\pi}{2} \tau(A, B)}{\ell \left( 1+p \right) \left( 1+p(A-B) \right) + \tau \sin \frac{\pi}{2} \tau(A, B)} \right) = -\frac{\pi}{2} \delta,
\]

where \( \lambda \) and \( \tau(A, B) \) are given by (4.1) and (4.2), respectively, which contradicts the assumption. This completes the proof of Theorem 3.

Taking \( q = 2, s = \lambda = \mu = A = 1, B = 0 \) and \( \alpha_1 = \alpha_2 = \beta = p \left( p \in \mathbb{N} \right) \) in Theorem 3, we have the following corollary:

**Corollary 3.** Let \( f(z) \in \sum_{p,m} \). If

\[
-\text{Re}\left\{ \frac{z(2(p+1)(2p+1)f'(z) + 4(1+p)zf''(z) + z^2 f'''(z))}{2p(2p+1)g(z) + 2(2p+1)zg'(z) + z^2 g''(z)} \right\} > \ell \ (0 \leq \ell < p),
\]

for some \( g(z) \in \sum_{p,m} \) satisfying the condition.
\[
\left| \frac{z(2(p+1)(2p+1)g'(z) + 4(p+1)zg''(z) + z^2g''''(z))}{2p(2p+1)g(z) + 2(2p+1)zg'(z) + z^2g''(z)} + p \right| < p.
\]

then

\[
-\Re\left\{ \frac{z((2p+1)f'(z) + zf''(z))}{zf'(z) + 2pf(z)} \right\} > \ell.
\]

Taking \( q = 2, s = \delta = A = 1, \lambda = 2, B = 0, \alpha_1 = p + 1 \) and \( \alpha_2 = \beta_1 = p \) \((p \in \mathbb{N})\) in Theorem 3, we have the following corollary:

**Corollary 4.** Let \( f(z) \in \sum_{p,m} \). If

\[
-\Re\left\{ \frac{z((2p+1)f'(z) + zf''(z))}{zf'(z) + 2pf(z)} \right\} > \ell \quad (0 \leq \ell < p),
\]

for some \( g(z) \in \sum_{p,m} \) satisfying the condition

\[
\left| \frac{z(2((p+1)(2p+3)g'(z) + 2(2p+3)zg''(z) + z^2g''''(z))}{2(p+1)(2p+1)g(z) + 4(p+1)zg'(z) + z^2g''(z)} + p \right| < p,
\]

then

\[
-\Re\left\{ \frac{z((2p+1)f'(z) + zf''(z))}{(2p+1)g(z) + zg'(z)} \right\} > \ell.
\]

Taking \( q = 2, s = 1, \lambda = \eta - p, m = 1 - p, \alpha_1 = n + p \) and \( \alpha_2 = \beta_1 = \eta \) \((\eta > 0, n > -p, p \in \mathbb{N})\) in Theorem 3, we have the following corollary:

**Corollary 5.** Let \( f(z) \in \sum_p \), \( 0 < \delta \leq 1 \), \( 0 \leq \ell < p \) and

\[
\eta \geq \frac{p(A - B)}{1 + B} \quad (-1 < B < A \leq 1; p \in \mathbb{N}).
\]

If

\[
\left| \text{arg} \left( -\frac{z(I_{n+p-1,q+1}f(z))}{I_{n+p-1,q+1}g(z)} - \ell \right) \right| < \frac{\pi}{2} \delta,
\]

for some \( g(z) \in \sum_{p,m} \) satisfying the condition

\[
-\frac{z(I_{n+p-1,q}g(z))}{I_{n+p-1,q}g(z)} < p \frac{1 + Az}{1 + Bz},
\]

then
where \( \tau(0 < \tau \leq 1) \) is the solution of the equation (4.1) with \( \lambda = \eta - p \) \( (\eta > 0) \).

Taking \( p = 1 \) in Corollary 5, we have the following corollary:

**Corollary 6.** Let \( f(z) \in \sum \), \( 0 < \delta \leq 1 \), \( 0 \leq \ell < 1 \) and

\[
\eta \geq \frac{A - B}{1 + B} \quad (-1 < B < A \leq 1).
\]

If

\[
\left| \arg \left( -\frac{z(I_{n+1,q}f(z))}{I_{n+1,q}g(z)} - \ell \right) \right| < \frac{\pi}{2} \delta,
\]

for some \( g(z) \in \sum \) satisfying the condition

\[
-\frac{z(I_{n,q}g(z))}{I_{n,q}g(z)} < \frac{1 + Az}{1 + Bz},
\]

(4.7)

then

\[
\left| \arg \left( -\frac{z(I_{n,q}f(z))}{I_{n,q}g(z)} - \ell \right) \right| < \frac{\pi}{2} \tau,
\]

where \( \tau \in Q \cap [0, 1] \) is the solution of the equation (4.1) with \( \lambda = \eta - 1 \) \( (\eta > 0) \).

The proof of the next theorem is akin to that of Theorem 3 and so, we omit it.

**Theorem 4.** Let \( f(z) \in \sum_{p,m} \), \( 0 < \delta \leq 1 \), \( \ell > p \) and

\[
\lambda \geq \frac{p(A - B)}{1 + B} - p \quad (-1 < B < A \leq 1; p \in \mathbb{N}).
\]

If

\[
\left| \arg \left( \frac{z(M_{p,q,s}^{j+1,m}(\alpha) f(z))}{M_{p,q,s}^{j+1,m}(\alpha) g(z)} + \ell \right) \right| < \frac{\pi}{2} \delta
\]

for some \( g(z) \in \sum_{p,q,s}^{j+1,m}(\alpha; A,B) \), then

\[
\left| \arg \left( \frac{z(M_{p,q,s}^{j,m}(\alpha) f(z))}{M_{p,q,s}^{j,m}(\alpha) g(z)} + \ell \right) \right| < \frac{\pi}{2} \tau,
\]
where $\tau(0 < \tau \leq 1)$ is the solution of equation (4.1).

Theorem 5. Let $f(z) \in \sum_{p,m,}$, $0 < \delta \leq 1$, $0 \leq \ell < p$ and

$$
\alpha_1 \geq \frac{p(A - B)}{1 + B} (-1 < B < A \leq 1; p \in \mathbb{N}).
$$

If

$$
\left| \arg \left( -\left( \frac{z(M_{p,q,s}^{*,m}(\alpha_1) f(z))}{M_{p,q,s}^{*,m}(\alpha_1) g(z)} - \ell \right) \right) \right| < \frac{\pi}{2} \delta,
$$

for some $g(z) \in \sum_{p,q,s}^{*,m} (\alpha_1; A, B)$, then

$$
\left| \arg \left( -\left( \frac{z(M_{p,q,s}^{*,m}(\alpha_1 + 1) f(z))}{M_{p,q,s}^{*,m}(\alpha_1 + 1) g(z)} - \ell \right) \right) \right| < \frac{\pi}{2} \tau,
$$

where $\tau(0 < \tau \leq 1)$ is the solution of equation (4.1).

Proof. Let

$$
X(z) = \frac{1}{p - \ell} \left( -\left( \frac{z(M_{p,q,s}^{*,m}(\alpha_1 + 1) f(z))}{M_{p,q,s}^{*,m}(\alpha_1 + 1) g(z)} - \ell \right) \right),
$$

(4.8)

where $X(z)$ is analytic with $X(0) = 1$. Using (1.7), we have

$$
-(p - \ell)X(z) - \ell[M_{p,q,s}^{*,m}(\alpha_1 + 1) g(z) - \alpha_1 M_{p,q,s}^{*,m}(\alpha_1) f(z) - (\alpha_1 + p) M_{p,q,s}^{*,m}(\alpha_1) f(z)]
$$

(4.9)

Differentiating (4.9) with respect to $z$ and multiplying by $z$, we obtain

$$
-(p - \ell)zX'(z)M_{p,q,s}^{*,m}(\alpha_1 + 1) g(z) + [-(p - \ell) X(z) - \ell] z(M_{p,q,s}^{*,m}(\alpha_1 + 1) g(z))'
$$

$$
= \alpha_1 z(M_{p,q,s}^{*,m}(\alpha_1) f(z))' - (\alpha_1 + p) z(M_{p,q,s}^{*,m}(\alpha_1) f(z))'.
$$

(4.10)

Then, by using (4.8), (4.9) and (4.10), we have

$$
\frac{1}{p - \ell} \left( -\left( \frac{z(M_{p,q,s}^{*,m}(\alpha_1) f(z))}{M_{p,q,s}^{*,m}(\alpha_1) g(z)} - \ell \right) \right) = X(z) + \frac{zX'(z)}{-J(z) + \alpha_1 + p},
$$

where

$$
J(z) = -\left( \frac{z(M_{p,q,s}^{*,m}(\alpha_1 + 1) g(z))}{M_{p,q,s}^{*,m}(\alpha_1 + 1) g(z)} \right).
$$
The remaining part of the proof is similar to that of Theorem 3 and so we omit it.

Taking \( q = 2, s = 1, \lambda = \eta - p, \quad m = 1 - p, \alpha_i = n + p \) and \( \alpha_z = \beta_i = \eta \quad (\eta > 0, n > -p, p \in \mathbb{N}) \) in Theorem 5, we have the following corollary:

**Corollary 7.** Let \( f(z) \in \Sigma_{p,m}, \quad 0 < \delta \leq 1, 0 \leq \ell < p \) and

\[
n \geq \frac{p(A-B)}{1+B} - p \quad (-1 < B < A \leq 1; p \in \mathbb{N}).
\]

If

\[
\left| \arg \left( \frac{-z(I_{n+p,q}f(z))}{I_{n+p,q}g(z)} - \ell \right) \right| < \frac{\pi}{2} \delta,
\]

for some \( g(z) \in \Sigma_{p,m} \) satisfying (4.6), then

\[
\left| \arg \left( \frac{-z(I_{n+p,q}f(z))}{I_{n+p,q}g(z)} - \ell \right) \right| < \frac{\pi}{2} \tau,
\]

where \( \tau(0 < \tau \leq 1) \) is the solution of equation (4.1).

Taking \( p = 1 \) in Corollary 7, we have the following corollary:

**Corollary 8.** Let \( f(z) \in \Sigma, \quad 0 < \delta \leq 1, 0 \leq \ell < 1 \) and

\[
n \geq \frac{(A-B)}{1+B} - 1 \quad (-1 < B < A \leq 1).
\]

If

\[
\left| \arg \left( \frac{-z(I_{n,q}f(z))}{I_{n,q}g(z)} - \ell \right) \right| < \frac{\pi}{2} \delta,
\]

for some \( g(z) \in \Sigma \) satisfying (4.7), then

\[
\left| \arg \left( \frac{-z(I_{n+1,q}f(z))}{I_{n+1,q}g(z)} - \ell \right) \right| < \frac{\pi}{2} \tau,
\]

where \( \tau(0 < \tau \leq 1) \) is the solution of equation (4.1).

The proof of the next theorem is akin to that of Theorem 5 and so, we omit it.

**Theorem 6.** Let \( f(z) \in \Sigma_{p,m}, \quad 0 < \delta \leq 1, \quad \ell > p \) and
\[\alpha_i \geq \frac{p(A-B)}{1+B} \quad (-1 < B < A \leq 1; \, p \in \mathbb{N}).\]

If
\[
\left| \arg \left( \frac{z\left(\mathcal{M}^{\lambda, m}_{p,q,s}(\alpha_i) f(z)\right) + \ell}{\mathcal{M}^{\lambda, m}_{p,q,s}(\alpha_i) g(z)} \right) \right| < \frac{\pi}{2} \delta,
\]
for some \( g(z) \in \sum^{\lambda, m}_{p,q,s}(\alpha_i; A, B) \), then
\[
\left| \arg \left( \frac{z\left(\mathcal{M}^{\lambda, m}_{p,q,s}(\alpha_i + 1) f(z)\right) + \ell}{\mathcal{M}^{\lambda, m}_{p,q,s}(\alpha_i + 1) g(z)} \right) \right| < \frac{\pi}{2} \tau,
\]
where \( \tau(0 < \tau \leq 1) \) is the solution of equation (4.1).

**Theorem 7.** Let \( f(z) \in \sum_{p,m}, \, 0 < \delta \leq 1, \, 0 < \ell < p \) and
\[
\mu \geq \frac{p(A-B)}{1+B} \quad (-1 < B < A \leq 1; \, p \in \mathbb{N}).
\]

If
\[
\left| \arg \left( -\frac{z\left(\mathcal{M}^{\lambda, m}_{p,q,s}(\alpha_i) f(z)\right) - \ell}{\mathcal{M}^{\lambda, m}_{p,q,s}(\alpha_i) g(z)} \right) \right| < \frac{\pi}{2} \delta,
\]
for some \( g(z) \in \sum^{\lambda, m}_{p,q,s}(\alpha_i; A, B) \), then
\[
\left| \arg \left( -\frac{z\left(\mathcal{M}^{\lambda, m}_{p,q,s}(\alpha_i) F_{\mu,p}(f)(z)\right) - \ell}{\mathcal{M}^{\lambda, m}_{p,q,s}(\alpha_i) F_{\mu,p}(g)(z)} \right) \right| < \frac{\pi}{2} \tau,
\]
where \( \tau(0 < \tau \leq 1) \) is the solution of equation (4.1).

**Proof.** Let
\[
k(z) = \frac{1}{p-\ell} \left( -\frac{z\left(\mathcal{M}^{\lambda, m}_{p,q,s}(\alpha_i) F_{\mu,p}(f)(z)\right) - \ell}{\mathcal{M}^{\lambda, m}_{p,q,s}(\alpha_i) F_{\mu,p}(g)(z)} \right), \quad (4.11)
\]
where \( k(z) \) is analytic with \( k(0) = 1 \). Using (1.9), we have
\[
[-(p-\ell)k(z) - \ell] \mathcal{M}^{\lambda, m}_{p,q,s}(\alpha_i) F_{\mu,p}(g)(z) = \mu \mathcal{M}^{\lambda, m}_{p,q,s}(\alpha_i) f(z) - (\mu + p) \mathcal{M}^{\lambda, m}_{p,q,s}(\alpha_i) F_{\mu,p}(f)(z). \quad (4.12)
\]
Differentiating (4.12) with respect to \( z \) and multiplying by \( z \), we obtain
\[
-(p-\ell)zk'(z)\mathcal{M}^{\lambda, m}_{p,q,s}(\alpha_i) F_{\mu,p}(g)(z) + [-p-\ell]z(k'(z) - \ell)z(\mathcal{M}^{\lambda, m}_{p,q,s}(\alpha_i) F_{\mu,p}(g)(z))'.
\]
\[ = \mu(\mathbf{M}_{p,q}, (\alpha_i) f(z))' - (\mu + p) z(\mathbf{M}_{p,q,s}, (\alpha_i) F_{\mu,p} (f)(z))'. \]  

Then, by using (4.11), (4.12) and (4.13), we have

\[
\frac{1}{p - \ell} \left( \frac{z(\mathbf{M}_{p,q,s}, (\alpha_i) f(z))'}{\mathbf{M}_{p,q,s} (\alpha_i) g(z)} - \ell \right) = k(z) + \frac{z k(z)}{-\rho(z) + \mu + p},
\]

where

\[
\rho(z) = -\frac{z(\mathbf{M}_{p,q,s}, (\alpha_i) F_{\mu,p} (g)(z))'}{\mathbf{M}_{p,q,s} (\alpha_i) F_{\mu,p} (g)(z)}.
\]

The remaining part of the proof is similar to that of Theorem 3 and so we omit it.

The proof of the next theorem is akin to that of Theorem 7 and so, we omit it.

**Theorem 8.** Let \( f(z) \in \Sigma_{p,m}, \ 0 < \delta \leq 1, \ \ell > p \) and

\[ \mu \geq \frac{p(A - B)}{1 + B} \quad (-1 < B < A \leq 1; \ p \in \mathbb{N}). \]

If

\[
\left| \arg \left( \frac{z(\mathbf{M}_{p,q,s}, (\alpha_i) f(z))'}{\mathbf{M}_{p,q,s} (\alpha_i) g(z)} + \ell \right) \right| < \frac{\pi}{2} \delta
\]

for some \( g(z) \in \Sigma_{p,m} (\alpha_i; A, B) \), then

\[
\left| \arg \left( \frac{z(\mathbf{M}_{p,q,s}, (\alpha_i) F_{\mu,p} (f)(z))'}{\mathbf{M}_{p,q,s} (\alpha_i) F_{\mu,p} (g)(z)} + \ell \right) \right| < \frac{\pi}{2} \tau,
\]

where \( \tau(0 < \tau \leq 1) \) is the solution of equation (4.1).

**Remark 1.** Specializing the parameters \( p, q, s, m \) and \( \lambda \) in the above results, we obtain the results for the corresponding operators defined in the introduction.

**Remark 2.** (i) Putting \( q = 2, s = 1, m = 1 - p, \ \alpha_i = \beta_i = a \quad (a > 0, p \in \mathbb{N}) \) and \( \alpha_2 = 1 \) in Theorems 3 and 4, respectively, we obtain the results obtained by Aouf et al. [5];

(ii) Putting \( q = 2, s = 1, m = 0, \ \alpha_1 = a, \alpha_2 = p = 1 \) and \( \beta_i = a \quad (a > 0) \) in Theorems 3 and 4, respectively, we obtain the results obtained by Cho [8, Theorems 2.1 and 2.2, respectively];

(iii) Putting \( q = 2, s = 1, m = 1 - p, \ \alpha_1 = a, \alpha_2 = 1 \) and \( \beta_i = c \quad (a, c \in \mathbb{R} \setminus \mathbb{Z}_0, p \in \mathbb{N}) \) in the above results, we obtain the results obtained by Aouf et al. [5].
(iv) Putting $q = 2, s = 1, m = 1 - p$, $\alpha_1 = c, \alpha_2 = p + \lambda$ and $\beta_1 = a$ ($a \in \mathbb{R}, c \in \mathbb{R} \setminus \mathbb{Z}_0^-, \lambda > -p, p \in \mathbb{N}$) in Theorem 4, we obtain the results obtained by Lashin [14, Theorem 2.2].

References


