Uniqueness of Solution for Nonlinear Implicit Fractional Differential Equation

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ABSTRACT
We study the uniqueness of solution for nonlinear implicit fractional differential equation with initial condition involving Caputo fractional derivative. The technique used in our analysis is based on an application of Bihari and Medved inequalities.

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Caputo fractional derivative, fractional integral, Implicit fractional differential equation, Bihari and Medved inequalities.

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1. INTRODUCTION

Fractional calculus is generalization of ordinary differential equations and integrations to arbitrary non-integer orders. One can describe many physical phenomena arising in engineering, physics, economics and science more accurately through the fractional derivative formulation, see [4, 5, 7, 9, 11, 18, 21].

Within years the fractional calculus became a very attractive subject to mathematicians, and many different forms of fractional (i.e., noninteger) differential operators were reintroduced: the Grunwald–Letnikow, Riemann–Liouville, Hadamard, Caputo, Riesz (Hilfer 2000; Kilbas et al. 2006; Podlubny 1999; Samko et al. 1993) and the more recent notions of Cresson (2007), and Katugampola (2011) or variable order fractional operators introduced by Samko and Ross (1993). Several authors have studied the existence, uniqueness and qualitative properties of the initial value problems of fractional order by different techniques, refer [5, 7, 10, 14, 18, 21].

Recently, Pachpatte D. B. and Chinchane V. L. [1] have discussed the uniqueness of solution of fractional differential equation with the Riemann–Liouville derivative.

By motivation of the recent works [1, 15], we extend the results presented by Pachpatte D. B. and Chinchane V. L. for nonlinear implicit fractional differential equation with Caputo fractional derivative. In this paper we consider the initial value problem of the type

\[ ^cD^\alpha f(t) = f(t, x(t)), \quad x(0) = x_0 \in \mathbb{R}, \quad t \in [0, b] \tag{1.1} \]

where \(^cD^\alpha (0 < \alpha < 1)\) denotes the Caputo fractional derivative operator and \(f\) is real continuous valued function on \(J \times \mathbb{R} \rightarrow \mathbb{R}\) into \(\mathbb{R}\); \(\mathbb{R}\) denotes the real space.

The paper is organized as follows. In Section 2, some definitions, lemmas and preliminary results are introduced to be used in the sequel. Section 3 will deals the results and proofs of uniqueness problem of (1.1).
**Definition 2.5.** The (left-sided) Riemann–Liouville fractional derivative of order \( \alpha > 0 \) of the function \( f \in C^n_{a_1} \ (n \in \mathbb{N} \cup \{0\}) \), is given by:

\[
D^\alpha f(t) = \frac{d^n}{dt^n} I^{n-\alpha} f(t), \quad n - 1 < \alpha \leq n, \ n \in \mathbb{N}. \quad (2.3)
\]

In [15], the author have been studied the existence and uniqueness of the initial value problem (1.1)-(1.2), first, let us reduce the problem (1.1)-(1.2) into equivalent fractional integral equation, we obtain

\[
x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \sigma)^{\alpha-1} f(\sigma, x(\sigma)) \, d\sigma, \quad t > 0. \quad (2.4)
\]

The results of our work in the present paper differ substantially from the work of [8, 15] in terms of assumptions and methods of proof.

Also, in [17] Medved defined a special class of nonlinear function and developed a method to estimate solution for nonlinear integral inequalities with singular kernel. The class of function defined as follows:

**Definition 2.6.** [17] Let \( q > 0 \) be a real number and \( 0 < b \leq \infty \). The function \( w : \mathbb{R}^+ \rightarrow \mathbb{R} \) satisfies the following condition

\[
e^{-q} w(u(t)) \leq R(t)w(e^{-q}u(t)), \quad (2.5)
\]

for all \( u \in \mathbb{R}_+ \), \( t \in [0, b] \), where \( R(t) \) is a continuous, nonnegative function.

**Remark 2.1.** If \( w(u) = u^m, \ m > 0 \), then

\[
e^{-q} w(u(t))^m = e^{(m-1)q} w(e^{-q}u(t)), \quad (2.6)
\]

for any \( q > 1 \), i.e the condition (2.5) is satisfies with \( R(t) = e^{(m-1)q} R(t) So w satisfies the condition (2.5) with \( q > 1 \) and \( R(t) = 2\alpha^{-1}e^{qmt} \), see [12].

In [17], Medved introduced the following inequalities which are the best widely and known in the study of many qualitative properties of solution of differential equations.

**Lemma 2.1.** Let \( 0 \leq T \leq \infty \), \( u(t), b(t), a(t), a'(t) \in C([0, T], \mathbb{R}^+); w \in C(\mathbb{R}^+, \mathbb{R}) \) be nondecreasing function, \( w(0) = 0, w(u) > 0 \) on \( (0, T) \), and

\[
u(t) \leq a(t) + \int_0^t (t-s)^{\beta-1} b(s) w(u(s)) ds, \quad (2.7)
\]

for \( t \in [0, T] \) where \( \beta > 0 \) is constant. Then following hold:

(i). Suppose \( \beta > \frac{1}{2} \), and if \( w \) satisfies the condition (2.5) with \( q = 2 \), then

\[
u(t) \leq e^{\Gamma(\Omega^{-1}[\Omega(2a(t)^2) + g_1(t)])} \int_0^t \frac{d\nu}{w(\nu)}, \quad (2.8)
\]

for \( t \in [0, T_1] \), where

\[
g_1(t) = \frac{\Gamma(2\beta - 1)}{\Gamma(\alpha - 1)} \int_0^t R(t) b(s)^2 ds, \quad (2.9)
\]

where \( \Gamma \) is gamma function, \( \Omega(\nu) = \int_{\nu}^{\infty} \frac{ds}{w(s)^2}; \nu > 0 \), \( \Omega^{-1} \) is the inverse of \( \Omega \), and \( t \in \mathbb{R}_+ \) is such that \( \Omega(2a(t)^2) + g_1(t) \in \text{Dom}(\Omega^{-1}) \) for all \( t \in [0, T_1] \).

(ii). Let \( \beta = \frac{1}{2} \), and \( w \) satisfies the condition (2.5) with \( q = z = 2 \), where \( z = \frac{1+1}{\beta} \), i.e. \( \beta = \frac{1}{z+1} \). Let \( \Omega, \Omega^{-1} \) be as in part (i).

Then

\[
u(t) \leq e^{\Gamma(\Omega^{-1}[\Omega(2\alpha-a(t)^2) + g_2(t)])} \int_0^t \frac{d\nu}{w(\nu)^z}, \quad (2.10)
\]

for \( t \in [0, T_1] \), where

\[
g_2(t) = 2^{z-1} K^2 \int_0^t R(t) b(s)^2 ds, \quad (2.11)
\]

\[
K = \frac{\Gamma(1-\alpha/p)^2}{\Gamma(1-\beta/p)}, \quad \alpha = \frac{z+1}{z+2}, \quad p = \frac{z+2}{z+1} \quad (2.12)
\]

and \( T_1 \in \mathbb{R}_+ \) is such that \( \Omega(2^{z-1}a(t)^2) + g_2(t) \in \text{Dom}(\Omega^{-1}) \) for all \( t \in [0, T_1] \).

**Lemma 2.2.** Let \( 0 \leq T \leq \infty \), \( u(t), b(t), a(t), a'(t) \in C([0, T], \mathbb{R}^+); w \in C(\mathbb{R}^+, \mathbb{R}) \) and

\[
u(t) \leq a(t) + \int_0^t (t-s)^{\beta-1} b(s) w(u(s)) ds, \quad (2.13)
\]

for \( t \in [0, T] \) where \( \beta > 0 \) is constant. Then following hold:

(i). Suppose \( \beta > \frac{1}{2} \), then
Using above estimation in (3.5), we get
\[ u(t) \leq \left( \frac{\sqrt{2}}{\alpha} \right) \left( 1 + \frac{1}{1 - \frac{1}{\beta}} \right) \int_0^t (t-s)^{\frac{\beta-1}{\beta}} b(s) ds + t. \] (2.16)

But by hypothesis (3.1), for any \( t \in [0,b] \) and any \( x, y \in \mathbb{R} \),
\[ |D^\alpha(x(t) - y(t))| = |f(t,x(t),\; D^\alpha x(t)) - f(t,y(t),\; D^\alpha y(t))| \leq M \Phi(|x(t) - y(t)|) + L |D^\alpha(x(t) - y(t))|. \]

This implies
\[ |D^\alpha(x(t) - y(t))| \leq \frac{ML}{1-L} \Phi(|x(t) - y(t)|). \] (3.6)

Using above estimation in (3.5), we get
\[ |x(t) - y(t)| \leq \epsilon + \left( \frac{1}{\Gamma(\alpha)} \right) \int_0^t (t-s)^{\frac{\alpha-1}{\alpha}} \left[ M \Phi(|x(s) - y(s)|) + \frac{ML}{1-L} \Phi(|x(s) - y(s)|) \right] ds \]

and
\[ |y(t)| \leq \gamma + \left( \frac{1}{\Gamma(\alpha)} \right) \int_0^t (t-s)^{\frac{\alpha-1}{\alpha}} \left[ M \Phi(|x(s) - y(s)|) + \frac{ML}{1-L} \Phi(|x(s) - y(s)|) \right] ds \] (2.17)
\[ \leq \epsilon + \frac{1}{\Gamma(a)} \int_0^1 (t-s)^{a-1} M \left[ 1 + \frac{L}{1-L} \right] \Phi((x(s) - y(s))) ds \]
\[ \leq \epsilon + \frac{1}{\Gamma(a)} \int_0^1 (t-s)^{a-1} \frac{M}{1-L} \Phi((x(s) - y(s))) ds. \] 

(3.7)

Now an application of Lemma 2.3 to (3.7) which yields
\[ |x(t) - y(t)| < \Phi^{-1} \left[ \Phi(\epsilon) + \frac{M}{\Gamma(a)(1-L)} \int_0^b (t-s)^{a-1} ds \right] \]
\[ < \Phi^{-1} \left[ \Phi(\epsilon) + \frac{M}{\Gamma(a)(1-L)} \right] \]
\[ < \Phi^{-1} \left[ \Phi(\epsilon) + \frac{M}{\Gamma(a)(1-L)} \left( \frac{b^a}{a} - \frac{t-b}{a} \right)^a \right] \]
\[ < \Phi^{-1} \left[ \Phi(\epsilon) + \frac{M}{\Gamma(a)(1-L)} \left( \frac{b^a}{a} - \frac{t-b}{a} \right)^a \right], \] 

(3.8)

where \( \Phi(x) \) is primitive for \( \frac{1}{\Phi(x)} \). We shall prove that the right-hand side of (3.8) tends toward zero as \( \epsilon \to 0 \). As \( |x(t) - y(t)| \) is independent of \( \epsilon \), it follows that \( x(t) = y(t) \), which we need. Let us remark that condition (3.2) implies \( \Phi(\epsilon) \to -\infty \) for \( \epsilon \to 0 \), no matter how we choose the primitive of \( \frac{1}{\Phi(x)} \). Thus \( \Phi^{-1}(\epsilon) \to 0 \) as \( x \to -\infty \). Consequently, \( \epsilon \to 0 \) in the inequality (3.8), the right-hand side tends toward zero. This completes the proof of the theorem.

**Theorem 3.2** If the function \( f \) is continuous and satisfies the condition
\[ |f(t,x,y) - f(t,x',y)| \leq M|x - x'| + L|y - y'|, \] 

for some positive constant \( M \) and \( L \in (0,1) \) then the initial value problem (3.1)-(3.2) has unique solution in the interval \( J \).

**Proof.** Assume that there exists two solutions \( x(t) \) and \( y(t) \) of (1.1)-(1.2). Then we have
\[ x(t) = x_0 + \frac{1}{\Gamma(a)} \int_0^t (t-s)^{a-1} f(s,x(s), \quad ^CD^a x(s)) ds \] 
\[ y(t) = y_0 + \frac{1}{\Gamma(a)} \int_0^t (t-s)^{a-1} f(s,y(s), \quad ^CD^a y(s)) ds. \] 

(3.10)

(3.11)

Therefore, using these (3.10), (3.11) and hypothesis (3.9), we have
\[ |x(t) - y(t)| \leq \frac{1}{\Gamma(a)} \int_0^t \left( (t-s)^{a-1} \right) f(s,x(s), \quad ^CD^a x(s) - f(s,y(s), \quad ^CD^a y(s)) \right) ds \]
\[ \leq \epsilon + \frac{1}{\Gamma(a)} \int_0^t (t-s)^{a-1} M((x(s) - y(s)) + L |^CD^a(x(s) - y(s))|) ds \] 

(3.12)

But by hypothesis (3.9) for any \( t \in [0,b] \) and any \( x,y \in \mathbb{R} \),
\[ |^CD^a(x(t) - y(t))| = |f(t,x(t), \quad ^CD^a x(t)) - f(t,y(t), \quad ^CD^a y(t))| \]
\[ \leq M(|x(t) - y(t)|) + L |^CD^a(x(t) - y(t))|, \]

This implies
\[ |^CD^a(x(t) - y(t))| \leq \frac{M}{1-L} \epsilon \]

(3.13)

Using (3.13) in (3.12), we obtain
\[ |x(t) - y(t)| \leq \epsilon + \frac{1}{\Gamma(a)} \int_0^t (t-s)^{a-1} \left[ M(|x(s) - y(s)|) + \frac{ML}{1-L} (|x(s) - y(s)|) \right] ds \]
\[ \leq \epsilon + \frac{1}{\Gamma(a)} \int_0^t (t-s)^{a-1} \left[ M + \frac{ML}{1-L} \right] (|x(s) - y(s)|) ds \]
\[ \leq \epsilon + \frac{1}{\Gamma(a)} \int_0^t (t-s)^{a-1} \left[ \frac{M}{1-L} (|x(s) - y(s)|) \right] ds \]
\[ \leq \epsilon + \frac{1}{\Gamma(a)} \int_0^t (t-s)^{a-1} \frac{M}{1-L} (|x(s) - y(s)|) ds. \] 

(3.14)

Now, (a) suppose that \( \alpha > \frac{1}{2} \), then applying Lemma 2.2 (i) to (3.14), we have
\[ |x(t) - y(t)| < \sqrt{2} \epsilon \exp \left[ \frac{2\sqrt{2\alpha} - 1}{4\alpha} \int_0^t \left( \frac{1}{\Gamma(a)(1-L)} \right)^2 ds + t \right] \]
\[ < \sqrt{2} \epsilon \exp \left[ \frac{2\sqrt{2\alpha} - 1}{4\alpha} \frac{M}{(1-L)^2} \right] \]
\[ \leq \sqrt{2} \epsilon \exp \left[ \frac{2\sqrt{2\alpha} - 1}{4\alpha} \frac{M}{(1-L)^2} t + t \right], \] 

(3.15)

for \( t \in J \). Since \( \epsilon \) was arbitrary, as \( \epsilon \to 0 \) the inequality (3.13) implies that \( x(t) = y(t) \) on \( J \).
(b) Let $\alpha > \frac{1}{z+1}$ for some $z \geq 1$. Then by Lemma 2.2(iii) to (3.14), again we have,

$$|x(t) - y(t)| < (2^{q-1})^2 \varepsilon \exp \left[ \frac{2^{q-1}}{q} \int_0^t \frac{1}{(\alpha - 1)} M \left( \frac{1}{\alpha} \right) \frac{K_2^1}{(\alpha - 1)} ds + t \right]$$

$$< (2^{q-1})^2 \varepsilon \exp \left[ \frac{2^{q-1}}{q} K_2^1 \frac{1}{\Gamma(\alpha - 1)} M \right] t + t$$ \hspace{1cm} (3.16)

for $t \in [0, T)$ where $K_i$ is defined by (2.12). Since $\varepsilon$ was arbitrary in (3.16), it implies that $x(t) = y(t)$ as $\varepsilon \to 0$. This completes the proof of the theorem.

REFERENCES


