The global attractors and their Hausdorff and fractal dimensions
estimation for the higher-order nonlinear Kirchhoff-type equation with
nonlinear strongly damped terms*

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Abstract
In this paper, we study the long time behavior of solution to the initial boundary value problems for higher -order kirchhoff-type equation with nonlinear strongly dissipation:

\[ u_t + (-\Delta)^m u_t + \left( \int_{\Omega} |\nabla u|^p \right)^\gamma (-\Delta)^m u + h(u) = f(x). \]

At first, we prove the existence and uniqueness of the solution by priori estimate and Galerkin method then we establish the existence of global attractors, at last, we consider that estimation of upper bounds of Hausdorff and fractal dimensions for the global attractors are obtain.

Keywords: Higher-order nonlinear Kirchhoff wave equation; The existence and uniqueness; The Global attractors; Hausdorff dimensions; Fractal dimensions

1 Introduction
In this paper, we concerned with the long time behavior of solution to the initial boundary value problems for Higher-order Kirchhoff-type equation with nonlinear strongly dissipation:

\[ u_t + (-\Delta)^m u_t + \left( \int_{\Omega} |\nabla u|^p \right)^\gamma (-\Delta)^m u + h(u) = f(x) \quad (1.1) \]

\[ u(x,t) = 0, \quad \frac{\partial^i u}{\partial v^i} = 0, \quad i = 1, 2 \cdots, m - 1, x \in \partial \Omega, t > 0, \quad (1.2) \]

\[ u(x) = u_0, \quad u_i = u_i(x), \quad x \in \partial \Omega \quad (1.3) \]

Where \( \Omega \subset \mathbb{R}^n \) is bounded open domain with smooth boundary; \( v \) is the outer norm vector; \( m > 1 \) is a positive integer, and \( q > 0 \) is a positive constants, \( h(u) \) is a nonlinear forcing, \( (-\Delta)^m u \) is a strongly dissipation.

There have been many researches on the well-positive and the longtime dynamics for Kirchhoff equation. We can see[1-6], FUCAI Li [5] deals with the higher-order kirchhoff-type equation with nonlinear dissipation:

\[ u_t + \left( \int_{\Omega} |\nabla u|^p \right)^\gamma (-\Delta)^m u + |u|^p u = |u|^p u, \quad x \in \Omega, t > 0. \quad (1.4)\]
\[ u(x,t) = 0, \quad \frac{\partial^i u}{\partial v^i} = 0, \quad i = 1,2 \cdots , m-1, \quad x \in \partial \Omega, \quad t > 0. \tag{1.5} \]

\[ u(x) = u_0, \quad u_i(x) = u_i(x), \quad x \in \partial \Omega. \tag{1.6} \]

In a bounded domain, where \( m > 1 \) is a positive integer, \( p, q, r > 0 \) are positive constants and obtain that the solution exists globally if \( p \leq r \), while if \( p > \max\{ r, 2q \} \), then for any initial data with negative initial energy, the solution blows up at finite time in \( L^{p+2} \) norm.

Yang Zhijian, Wang Yunqing [6] also studied the global attractor for the Kirchhoff type equation with a strong dissipation:

\[ u_x - M \left( \left\| \Delta u \right\| \right)^2 \Delta u - \Delta u_i + h(u_i) + g(u) = f(x) \quad \text{in} \quad \Omega \times \mathbb{R}^+, \tag{1.7} \]

\[ u(x,t) \big|_{\partial \Omega} = 0, \quad t > 0, \tag{1.8} \]

\[ u(x,0) = u_0(x), \quad u_i(x,0) = u_i(x), \quad x \in \Omega. \tag{1.9} \]

Where \( M(s) = 1 + s^{\frac{m}{2}}, \quad 1 \leq m \leq \frac{4}{N-2}, \quad \Omega \) is a bounded domain in \( \mathbb{R}^N \), with smooth boundary \( \partial \Omega \), \( h(x) \) and \( g(s) \) are nonlinear functions, and \( f(x) \) is an external force term. It proves that the relative continuous semigroup \( S(t) \) possesses in the phase space with low regularity a global attractor which is connected.

Yang Zhijian, Cheng Jianling [7] studies the asymptotic behavior of solutions to the Kirchhoff-type equation:

\[ u_x - M \left( \left\| \Delta u \right\| \right)^2 \Delta u - \Delta u_i + h(u_i) + g(u) = f(x) \quad \text{in} \quad \Omega \times \mathbb{R}^+, \tag{1.10} \]

\[ u(x,t) \big|_{\partial \Omega} = 0, \quad t > 0, \tag{1.11} \]

\[ u(x,0) = u_0(x), \quad u_i(x,0) = u_i(x), \quad x \in \Omega. \tag{1.12} \]

They prove that the related continuous semigroup \( S(t) \) possesses in phase space \( \mathcal{X} = (H^2(\Omega) \cap H^1_0) \times H^1_0(\Omega) \) a global attractor. At the end of the paper, an example is shown, which indicates the existence of nonlinear functions \( g(x, u) \) and \( h(u_i) \).

Recently, Meixia Wang, Cuicui Tian, Guoguang Lin [8] studied the global attractor and dimension for a 2D generalized Anisotropy Kuramoto-Sivashinsky equation:
\[ u + \alpha \Delta^2 u + \gamma u + (\varphi(u))_{xx} + (g(u))_{yy} = f(x), \quad (x,y) \in \Omega \subset R^2, \quad (1.13) \]

\[ u(x,y,t)|_{t=0} = u_0(x), \quad (x,y) \in \Omega \subset R^2, \quad (1.14) \]

\[ u(x,y,t)|_{\partial \Omega} = 0, \quad \Delta u(x,y,t)|_{\Omega} = 0, \quad (x,y) \in \Omega \subset R^2, \quad (1.15) \]

Where \( \Omega \subset R^2 \) is bounded set; \( \partial \Omega \) is the bound of \( \Omega \); \( \varphi(u) \) and \( g(u) \) are considered as smooth function of \( u(x,y,t) \). Under the existence of the global solution, it proves that the global attractor and Hausdorff dimension and fractal dimension.

The paper is arranged as follows. In section 2, we state some preliminaries under the assume of Lemma 1 and Lemma 2, we get the existence and uniqueness of solution; in section 3, we obtain the global attractors for the problems (1.1)-(1.3); in section 4, we consider that the global attractor of the above mentioned problems (1.1)-(1.3) has finite Hausdorff dimension and fractal dimensions.

2 Preliminaries

For convenience, we denote the norm and scalar product in \( L^2(\Omega) \) by \( \| \cdot \|_2 \) and \( (\cdot,\cdot) \); \( f = f(x) \),

\[ H^k = H^k(\Omega), \quad H_0^k = H_0^k(\Omega), \quad H^{-k} = H^{-k}(\Omega), \quad \| \cdot \|_{k} = \| \cdot \|_{L^2_k}, \quad C, (i = 0, \ldots, 8), \kappa \] are constants,

\[ K_0 \geq \max \left\{ \frac{q \varepsilon}{q+1}, \frac{q}{q+1}, \frac{2 \alpha \varepsilon^2}{q+1}, \frac{4 C \varepsilon (2q - 2p + 2)}{2p + 2} \right\}. \]

In this section, we present some materials needed in the proof of our results, state a global existence result, and prove our main result. For this reason, we assume that and notations needed in the proof of our results. For this reason, we assume that

\[ (G_1) \quad \text{there exist} \]

\[ 0 < \delta < \frac{1}{2}, \quad \| h(u) \|_{L^2} \leq C_0 (h(u),u)^{1-\delta}, \quad \forall u \in H_0^n, \]

\[ (G_2) \quad \text{there exist constant} \]

\[ 0 < \sigma < 1, \quad \| h(v) \| \leq C_1 (R)(1 + \| \Delta v \|)^{1-\sigma}, \quad \forall v \in H^1 \cap H_0^n, \quad \| \cdot \| \leq R, \]

\[ (G_3) \quad \| h'(s) \| \leq C_2 \]

Lemma 1. Assume \( (G_1) \) hold, and \( (u_0,u_1) \in H_0^n(\Omega) \times L^2(\Omega), f \in L^2(\Omega), v = u_t + eu \), then the solution
\[(u, v) \in H^m(\Omega) \times L^2(\Omega),\text{ and}
\]
\[
\left\| \frac{\partial^m u}{\partial x^m} \right\|_{L^2}^2 + \left\| \frac{\partial^m v}{\partial x^m} \right\|_{L^2}^2 \leq \frac{W(0)}{1 - \varepsilon} \left( 1 - e^{-\alpha t} \right) - K_0 + \frac{q}{q + 1} \frac{e^{-\alpha t}}{1 - \varepsilon}
\]
(2.1) Where \( v = u_0 + \varepsilon u \),

\[
0 < \varepsilon < \min \{ 1, \frac{\sqrt{1 + 4 \lambda_i^m - 1}}{2}, \frac{q + 1}{4 \alpha C_p}, \frac{q + 1}{2 \alpha} \} \]

\[
W(0) = \left\| \frac{\partial^m u_0}{\partial x^m} \right\|_{L^2}^2 + \frac{q}{q + 1} \left[ \left\| \frac{\partial^m u_0}{\partial x^m} \right\|_{L^2}^2 - K_0 \right] , \quad v_0 = u_0 + \varepsilon u_0,
\]

such that

\[
\left\| \frac{\partial^m u}{\partial x^m} \right\|_{L^2}^2 + \left\| \frac{\partial^m v}{\partial x^m} \right\|_{L^2}^2 \leq R_0 (t > t_1).
\]

Proof. Let \( v = u_0 + \varepsilon u \) we multiply \( v \) with both sides of equation (1.1) and obtain

\[
(u_0 + (-\Delta)^m u_0 + \left( \int_\Omega |\nabla^m u|^2 \right)^{\frac{m}{2}} (-\Delta)^m u + h(u_0), v) = (f(x), v),
\]
(2.2)

\[
(u_0, v) = (v_0 - \varepsilon u_0, v)
\]

\[
= (v, v) - \varepsilon (v - \varepsilon u, v)
\]

\[
= \frac{1}{2} \frac{d}{dt} \left[ \int_\Omega |v|^2 - \varepsilon (v - \varepsilon u, v) \right]
\]
(2.4)

\[
\geq \frac{1}{2} \frac{d}{dt} \left[ \int_\Omega |v|^2 - \varepsilon (v - \varepsilon u, v) \right] - \frac{\varepsilon^2}{2 \lambda_i} \left\| \frac{\partial^m u}{\partial x^m} \right\|_{L^2}^2 - \frac{\varepsilon^2}{2} \left\| \frac{\partial^m v}{\partial x^m} \right\|_{L^2}^2,
\]

\[
((-\Delta)^m u_0, v) = ((-\Delta)^m (v - \varepsilon u), v)
\]
(2.5)

\[
\geq \left\| \frac{\partial^m v}{\partial x^m} \right\|_{L^2}^2 - \frac{1}{2} \frac{d}{dt} \left\| \frac{\partial^m u}{\partial x^m} \right\|_{L^2}^2 - \varepsilon \left\| \frac{\partial^m v}{\partial x^m} \right\|_{L^2}^2
\]

\[
\left( \left\| \frac{\partial^m u}{\partial x^m} \right\|_{L^2}^2 (-\Delta)^m u_0, v \right) = \left( \left\| \frac{\partial^m u}{\partial x^m} \right\|_{L^2}^2 (-\Delta)^m u_0, u_0 + \varepsilon u \right)
\]
(2.6)

\[
= \frac{1}{2q + 1} \frac{d}{dt} \left\| \frac{\partial^m u}{\partial x^m} \right\|_{L^{q+2}}^{q+2} + \varepsilon \left\| \frac{\partial^m u}{\partial x^m} \right\|_{L^{q+2}}^{q+2}
\]

\[
(h(u_0), v) = (h(u_0), u_0 + \varepsilon u)
\]
(2.7)
\[ (\varepsilon(h(u), u) + (h(u), \varepsilon u). \]

\[ e \|h(u), u\| \leq \|h(u)\|_{u} - \|\nabla^a u\| \]

\[ \leq e \epsilon C(h(u), u)^{1/2} \|\nabla^a u\| \]

\[ \leq \frac{1}{2} (h(u), u) + C \epsilon^2 \|\nabla^a u\|^p \quad (2.8) \]

\[ \leq \frac{1}{2} (h(u), u) + C \epsilon^2 \|\nabla^a u\|^p. \]

\[ (f(x), v) \leq \|f\|_{L^p} \leq \frac{\epsilon^2}{2} \|v\|^p + \frac{1}{2 \epsilon^2} \|f\|^2 \quad (2.9) \]

For above, we have

\[ \frac{1}{2} \frac{d}{dt} \left( \|v\|^2 + \frac{1}{q+1} \|\nabla^a u\|^{q+2} - \epsilon \|\nabla^a u\|^q \right) + \frac{1}{q+1} \|\nabla^a u\|^{q+2} - \epsilon \|\nabla^a u\|^q \]

\[ + \epsilon \|\nabla^a u\|^{q+2} - \left( \epsilon^2 + \frac{\epsilon^2}{2 \lambda_1} \right) \|\nabla^a u\|^q - C \epsilon^2 \|\nabla^a u\|^p \quad (2.10) \]

\[ \leq \frac{1}{2 \epsilon^2} \|f\|^2. \]

By using Poincare inequality, we obtain:

\[ \|\nabla^a u\| \geq \lambda_1 \|v\| \quad (2.11) \]

By using Young’s inequality, we obtain

\[ \|\nabla^a u\| \leq \frac{1}{q+1} \|\nabla^a u\|^{q+2} + \frac{q}{q+1}, \quad (2.12) \]

\[ - \epsilon \|\nabla^a u\| \geq - \frac{\epsilon}{q+1} \|\nabla^a u\|^{q+2} - \frac{q \epsilon}{q+1} \quad (2.13) \]

Then we have
\[ \left\| \mathbf{v} \right\| + \frac{1}{q+1} \left\| \nabla^n u \right\|^{q+2} - \frac{\varepsilon}{q+1} \left\| \nabla^n u \right\|^{q+2} + K_0 \geq 0. \quad (2.14) \]

Taking \( \alpha = 2 + \frac{1}{\lambda_1} \) and \( p \leq q + 1 \) using Young's inequality we obtain

\[ - \alpha \varepsilon^2 \left\| \nabla^n u \right\|^2 \geq - \frac{\alpha \varepsilon^2}{q+1} \left\| \nabla^n u \right\|^{q+2} - \frac{\alpha \varepsilon^2}{q+1}, \quad (2.15) \]

\[ \left\| \nabla^n u \right\|^p \leq \frac{2p}{2q+2} \left\| \nabla^n u \right\|^{q+2} + \frac{2q - 2p + 2}{2p + 2}, \quad (2.16) \]

\[ -2C_o \varepsilon^2 \left\| \nabla^n u \right\|^2 \leq \frac{4pC_o \varepsilon^2}{2q+2} \left\| \nabla^n u \right\|^{q+2} + \frac{2C_o \varepsilon^2 (2q - 2p + 2)}{2p + 2}. \quad (2.17) \]

So we obtain

\[ \frac{\varepsilon}{2} \left\| \nabla^n u \right\|^{q+2} - \alpha \varepsilon^2 \left\| \nabla^n u \right\| + \frac{K_0}{2} \geq \frac{\varepsilon}{2} \left( - \frac{\alpha \varepsilon^2}{q+1} \left\| \nabla^n u \right\|^{q+2} + \frac{K_0}{2} - \frac{\alpha \varepsilon^2}{q+1} \right) \geq 0. \quad (2.18) \]

\[ \frac{\varepsilon}{2} \left\| \nabla^n u \right\|^{q+2} - 2C_o \varepsilon^2 \left\| \nabla^n u \right\|^{q} + \frac{K_0}{2} \]

\[ \geq \frac{\varepsilon}{2} \left\| \nabla^n u \right\|^{q+2} - \frac{4C_o \varepsilon^2}{2q+2} \left\| \nabla^n u \right\|^{q+2} + \frac{K_0}{2} - \frac{2C_o \varepsilon^2 (2q - 2p + 2)}{2p + 2} \geq 0. \quad (2.19) \]

So we have

\[ \frac{d}{dt} \left( \left\| \mathbf{v} \right\| + \frac{1}{q+1} \left\| \nabla^n u \right\|^{q+2} - \varepsilon \left\| \nabla^n u \right\|^{q+2} + K_0 \right) + (2\lambda_1 + 2\varepsilon - 2C_o \varepsilon^2) \left\| \nabla^n u \right\|^{q+2} \]

\[ + \varepsilon \left\| \nabla^n u \right\|^{q+2} + K_0 \geq 0. \quad (2.20) \]

\[ \leq \frac{1}{\varepsilon^2} \left\| f \right\|^2 + 2K_0. \]

Assume next we take \( \alpha_0 = (2\lambda^n - 2\varepsilon - 2C_o \varepsilon^2, (q+1)\varepsilon, 1) \) so we can obtain

\[ \frac{d}{dt} \left( \left\| \mathbf{v} \right\| + \frac{1}{q+1} \left\| \nabla^n u \right\|^{q+2} - \varepsilon \left\| \nabla^n u \right\|^{q+2} + K_0 \right) \]

\[ + \alpha_0 \left( \left\| \mathbf{v} \right\| + \frac{1}{q+1} \left\| \nabla^n u \right\|^{q+2} - \varepsilon \left\| \nabla^n u \right\|^{q+2} + K_0 \right) \geq 0. \quad (2.21) \]
\[ \leq \frac{1}{\varepsilon} \| f \|^2 + 2K_0. \]

Then we have
\[
\frac{d}{dt} W(t) + \alpha_0 W(t) \leq C_3, \tag{2.22}
\]
where \( W(t) = \| v \|^2 + \frac{1}{q+1} \| v^* u \|^{q+2} - \varepsilon \| v^* u \|^2 + K_0 \), \( C_3 = \frac{1}{\varepsilon^2} \| f \|^2 + 2K_0 \), by using Gronwall inequality, we obtain
\[
W(t) \leq W(0) e^{-\alpha_0 t} + \frac{C_3 (1 - e^{-\alpha_0 t})}{\alpha_0}, \tag{2.23}
\]
From (2.12), we know
\[
\| v^* u \| - \frac{q}{q+1} \leq \frac{1}{q+1} \| v^* u \|^{q+2}. \tag{2.24}
\]
So
\[
\| v \|^2 + \frac{1}{q+1} \| v^* u \|^{q+2} - \varepsilon \| v^* u \|^2 + K_0 \geq \| v \|^2 + (1 - \varepsilon) \| v^* u \|^2 + K_0 - \frac{q}{q+1} \geq 0, \tag{2.25}
\]
From (2.23), we obtain
\[
\| v \|^2 + (1 - \varepsilon) \| v^* u \|^{q+2} + K_0 - \frac{q}{q+1} \leq W(0) e^{-\alpha_0 t} + \frac{C_3 (1 - e^{-\alpha_0 t})}{\alpha_0}, \tag{2.26}
\]
Then we have
\[
(1 - \varepsilon)(\| v \|^2 + \| v^* u \|^{q+2}) \leq W(0) e^{-\alpha_0 t} + \frac{C_3 (1 - e^{-\alpha_0 t})}{\alpha_0} - K_0 + \frac{q}{q+1}, \tag{2.27}
\]
so, we obtain
\[
\| (u, v) \|^2_{H^s + L^q} = \| v^* u \|^2 + \| v \|^2 \leq \frac{W(0) e^{-\alpha_0 t} + \frac{C_3 (1 - e^{-\alpha_0 t})}{\alpha_0} - K_0 + \frac{q}{q+1}}{1 - \varepsilon}. \tag{2.28}
\]
Then
\[
\lim_{t \to \infty} \| (u, v) \|^2_{H^s + L^q} \leq \frac{C_3 - K_0 + \frac{q}{q+1}}{1 - \varepsilon}. \tag{2.29}
\]
So, there exist \( R_0 \) and \( t_0 = t_0(\Omega) > 0 \), such that
Lemma 2. In addition to the assumptions \((G_2)\) hold, \(f \in H^n(\Omega)\), then the solution \((u, v)\) of the problems

\[(1.1)-(1.3)\] satisfies \((u, v) \in H^{2n}(\Omega) \times H^n(\Omega)\), and

\[
\left\| u, v \right\|_{H_0^{2n}} = \left\| (\Delta)^n u \right\| + \left\| \nabla^m v \right\| \leq \frac{Y(0)e^{-\beta \rho}}{\beta_1} + \frac{C_1(1 - e^{-\beta \rho})}{\beta_0 \beta_1} \tag{2.31}
\]

In where \(v = u + \rho u\), and \(Y(0) = \left( \left\| \nabla^m u \right\|^2 - \epsilon \right) (\Delta)^n u \right\| + \left\| \nabla^m v \right\| \) thus there exist \(R_1\) and \(t_2 = t_2(\Omega) > 0\), such that

\[
\left\| u, v \right\|_{H_0^{2n}} = \left\| (\Delta)^n u \right\| + \left\| \nabla^m v \right\| \leq R_1(t > t_1) \tag{2.32}
\]

Proof. Let \((\Delta)^n v = (\Delta)^n u + \rho (\Delta)^n u\), we multiply \((\Delta)^n v\) with both sides of equation \((1.1)\), and obtain

\[
(u_n + (\Delta)^n u_n, (\Delta)^n v) = (f(x), (\Delta)^n v), \tag{2.33}
\]

\[
(u_n, (\Delta)^n v) \geq \frac{1}{2} \frac{d}{dt} \left\| \nabla^m v \right\|^2 - \epsilon \left\| \nabla^m v \right\|^2 - \frac{\epsilon^2}{2\lambda_1} (\Delta)^n u \right\|^2 - \frac{\epsilon^2}{2} \left\| \nabla^m v \right\|^2 \tag{2.34}
\]

\[
((\Delta)^n u_n, (\Delta)^n v) = ((\Delta)^n (v - \eta u), (\Delta)^n v) \tag{2.35}
\]

\[
= \left\| (\Delta)^n v \right\|^2 - \epsilon \left\| (\Delta)^n u \right\|^2 - \frac{\epsilon^2}{2} \left\| (\Delta)^n u \right\|^2 \tag{2.36}
\]

\[
\left\| \nabla^m u \right\|^2 (\Delta)^n u_n, (\Delta)^n v) = \frac{1}{2} \frac{d}{dt} \left\| \nabla^m u \right\|^2 - \frac{1}{2} (\Delta)^n u \right\|^2 - \frac{1}{2} \left\| \nabla^m u \right\|^2 \tag{2.37}
\]

Form \((G_2)\), we have

\[
\left\| h(u_n) \right\| \leq C_1 \left( R_1 + \left\| (\Delta)^n u_n \right\| \right)^{2(1 - \sigma)} \tag{2.38}
\]

By using Young’s inequality
\[ \left\| h(u) \right\| \leq \frac{\sigma}{\mu} (C^2_1(R) \sigma (1 - \sigma) \mu \frac{1}{1 - \sigma} ((1 + \left\| (-\Delta)^{\alpha} u \right\|^{2(1 - \sigma)}) \frac{1}{1 - \sigma} , \quad (2.39) \]

and

\[ \left\| h(u) \right\| \leq C_s \left\| \frac{1}{4} \right\| (-\Delta)^{\alpha} v \right\|^{\frac{1}{4}} + \frac{\varepsilon^2}{4} \left\| (-\Delta)^{\alpha} u \right\|^{\frac{1}{4}} , \quad (2.40) \]

Where \( C_s \) is defined as

\[ C_s := \frac{\sigma}{\mu} (C^2_1(R) \sigma (1 - \sigma) \mu \frac{1}{1 - \sigma} , \]

we take proper \( \mu \) such that \( 4 (1 - \sigma) \mu \frac{1}{1 - \sigma} = \frac{1}{4} \),

\[ K_1 = C_s. \]

\[ \left\| f(x), (-\Delta)^{\alpha} v \right\| \leq \frac{\left\| \nabla f \right\|^2}{2 \varepsilon^2} + \frac{\varepsilon^2 \left\| \nabla v \right\|^2}{2} \quad (2.41) \]

From above, we have

\[ \frac{d}{dt} \left( \left\| \nabla v \right\|^2 + \left\| \nabla u \right\|^2 \left\| (-\Delta)^{\alpha} u \right\| - \varepsilon \left\| (-\Delta)^{\alpha} u \right\|^{\frac{1}{2}} \right) \right) + \left( \frac{3 \lambda_{\alpha}^n}{4} - 2 \varepsilon - 2 \varepsilon^2 \right) \left\| \nabla v \right\|^2 \]

\[ + \left( \frac{d}{dt} \left\| \nabla u \right\|^2 + 2 \varepsilon \left\| \nabla v \right\|^2 - \frac{9 \varepsilon^2}{4} - \frac{\varepsilon^2}{\lambda_{\alpha}^n} \left\| (-\Delta)^{\alpha} u \right\|^{\frac{1}{2}} \right) \]

\[ \leq \frac{1}{\varepsilon^2} \left\| \nabla f \right\|^2 + 2 K_1. \quad (2.42) \]

Then we take proper \( \varepsilon \), let \( \frac{3 \lambda_{\alpha}^n}{4} - 2 \varepsilon - 2 \varepsilon^2 \geq 0 \) and \( \left\| \nabla u \right\|^q - \varepsilon > 0 \), next we assume exist a positive constant \( K > 0 \), let \( K - 2 \varepsilon \geq 0 \), satisfies

\[ 0 \leq K \left( \left\| \nabla u \right\|^q - \varepsilon \right) \leq - \frac{d}{dt} \left\| \nabla u \right\|^q + 2 \varepsilon \left\| \nabla u \right\|^q - \frac{9 \varepsilon^2}{4} - \frac{\varepsilon^2}{\lambda_{\alpha}^n} \quad (2.43) \]

where \( C_b := \frac{9 \varepsilon^2}{4} - \frac{\varepsilon^2}{\lambda_{\alpha}^n} + K \varepsilon \) such that

\[ (K - 2 \varepsilon) \left\| \nabla u \right\|^q + \frac{d}{dt} \left\| \nabla u \right\|^q \leq C_b . \quad (2.44) \]

Multiplying (2.44) by \( e^{(K - 2 \varepsilon) t} \) then

\[ \left\| \nabla u \right\|^q \frac{dt}{dt} e^{(K - 2 \varepsilon) t} \right) + e^{(K - 2 \varepsilon) t} \frac{d}{dt} \left\| \nabla u \right\|^q \leq C_b e^{(K - 2 \varepsilon) t} , \quad (2.45) \]

we integrate (2.45) with respect to time \( t \) and get that
\[
\varepsilon < \| \nabla^\alpha u \|_{r}^2 \leq \frac{C_\alpha}{K - 2\varepsilon} (1 + \kappa e^{-(K - 2\varepsilon)}),
\]
(2.46)

So, (2.43) exists a positive constant \( K \).

Form above, we have

\[
\frac{d}{dt} (\| \nabla^\alpha v \|^2 + \| \nabla^\alpha u \|^2) - \varepsilon \|(-\Delta)^\alpha u \|^2 + \left( \frac{3\lambda}{4} - 2\varepsilon - 2\varepsilon^2 \right) \| \nabla^\alpha v \|^2
\]

\[+ K \left( \| \nabla^\alpha u \|^2 - \varepsilon \right) \|(-\Delta)^\alpha u \| \]
(2.47)

\[
\leq \frac{1}{\varepsilon^2} \| \nabla f \|^2 + 2K_1.
\]

Taking \( \beta_0 = \min\left\{ \frac{3\lambda}{4} - 2\varepsilon - 2\varepsilon^2, K \right\} \), \( C_4 = \frac{1}{\varepsilon^2} \| \nabla f \|^2 + 2K_1 \), then

\[
\frac{d}{dt} Y(t) + \beta_0 Y(t) \leq C_4,
\]
(2.48)

assumptions where \( Y(t) = \| \nabla^\alpha v \|^2 + \| \nabla^\alpha u \|^2 - \varepsilon \) \( \|(-\Delta)^\alpha u \|^2 \) by using Gr"{o}nwall inequality, then

\[
Y(t) \leq Y(0)e^{-\beta_0 t} + \frac{C_4}{\beta_0} (1 - e^{-\beta_0 t}).
\]
(2.49)

Let \( \beta_1 = \min\{1, \inf_{i \in \mathbb{N}} \| \nabla^\alpha u \|^2 - \varepsilon \} \), we get \( \beta_1 \left( \| \nabla^\alpha v \|^2 + \|(-\Delta)^\alpha u \|^2 \right) \leq Y(t) \).

so we get

\[
\| (u, v) \|_{H^{\alpha+}\Omega} = \|(-\Delta)^\alpha u \| + \| \nabla^\alpha v \| \leq \frac{\| Y(t) e^{-\beta_0 t} \|}{\beta_1} + \frac{C_4 (1 - e^{-\beta_0 t})}{\beta_0 \beta_1}.
\]
(2.50)

where \( Y(t) = (\|(-\Delta)^\alpha u \|^2 - \varepsilon ) + \| \nabla^\alpha v \|^2 + \| \nabla^\alpha u \|^2 \), \( u_0 = u_1 + \varepsilon u_\nu \) then

\[
\lim_{t \to \infty} \| (u, v) \|_{H^{\alpha+}\Omega} \leq \frac{C_4}{\beta_0 \beta_1}.
\]
(2.51)

So, there exist \( R_1 \) and \( t_1 = t_1(\Omega) > 0 \), such that

\[
\| (u, v) \|_{H^{\alpha+}\Omega} = \|(-\Delta)^\alpha u \| + \| \nabla^\alpha v \| \leq R_1(t > t_1).
\]
(2.52)

3. Global attractor

3.1 the existence and uniqueness of solution

**Theorem 3.1** Assume \((G_1), (G_2), (G_3)\), holds and Lemma1 Lemma2 holds; the problem (1.1)-(1.3) exists a unique
smooth solution

\[(u, v) \in L^\infty (\mathbb{R}^+); H^m \times H^m \]  \hspace{1cm} (3.1)

Proof. By the Galerkin method, Lemma 1 and Lemma 2, we can easily obtain the existence of solution. Next, we prove the uniqueness of Solutions in detail

Assume \( u, v \) are two solutions of the problems (1.1)-(1.3). Let \( w = u - v \), then

\[ w(0,0) = w_0, \quad w_t(0,0) = w_1(x) = 0 \]

Then two equations subtract and obtain

\[ w_t + (-\Delta)^m w_t + \nabla v_t \nabla (-\Delta)^m u - \nabla v_t \nabla (-\Delta)^m v + h(w_t) - h(v_t) = 0 \]  \hspace{1cm} (3.2)

By multiplying (3.2) by \( w_t \), we get

\[ (w_t + (-\Delta)^m w_t + \nabla v_t \nabla (-\Delta)^m u - \nabla v_t \nabla (-\Delta)^m v + h(w_t) - h(v_t), w_t) = 0 \]  \hspace{1cm} (3.3)

\[ (w_t, w_t) = \frac{1}{2} \frac{d}{dt} \|w_t\|^2 \]  \hspace{1cm} (3.4)

\[ (-\Delta)^m w_t, w_t) = \|\nabla w_t\|^2 \]  \hspace{1cm} (3.5)

\[ (\nabla u \nabla (-\Delta)^m u - \nabla v \nabla (-\Delta)^m v \nabla u, w_t) \]

\[ = (\nabla u \nabla (-\Delta)^m u - \nabla v \nabla (-\Delta)^m v + \nabla v u \nabla (-\Delta)^m v - \nabla v \nabla (-\Delta)^m v, w_t) \]

\[ = (\nabla u \nabla (-\Delta)^m u, w_t) + ((\nabla u \nabla (-\Delta)^m u - \nabla v \nabla (-\Delta)^m v, w_t) \]  \hspace{1cm} (3.6)

\[ \frac{1}{2} \frac{d}{dt} (\| \nabla u \nabla (-\Delta)^m u \|^2) - q \| \nabla u \nabla (-\Delta)^m u \|^2 - \| \nabla u \nabla (-\Delta)^m u \|^2 \]

\[ + \| \nabla v \nabla (-\Delta)^m v \|^2 \]

\[ \leq 2q \| \nabla u \nabla (-\Delta)^m u \|^2 + \theta ((\| \nabla v \nabla (-\Delta)^m u \|^2) \]  \hspace{1cm} (3.7)

According to Lemma 1, Lemma 2, we have

\[ 2q \| \nabla u \nabla (-\Delta)^m u \|^2 + \theta ((\| \nabla v \nabla (-\Delta)^m u \|^2) \leq C_\gamma, \]  \hspace{1cm} (3.8)
\[ q \| \nabla \nabla u \|^{q-1} \| \nabla \nabla u \| \leq C. \]

Then
\[ \left| \left( \| \nabla \nabla u \|^{q} - \| \nabla \nabla v \|^{q} \right) (-\Delta)^{m} v, w_{j} \right| \leq C_{2} \| \nabla \nabla w \| \| w_{j} \|. \] (3.9)

According to Young’s inequality, we get
\[ \left| \left( \| \nabla \nabla u \|^{q} - \| \nabla \nabla v \|^{q} \right) (-\Delta)^{m} v, w_{j} \right| \leq \frac{C_{2}}{2} \| \nabla \nabla w \|^{2} + \frac{C_{2}}{2} \| w_{j} \|. \] (3.10)

Form \((G_{j})\), we have
\[ \left| h(u_{j}) - h(v_{j}), w_{j} \right| = \left| h'(w)w_{j} \right| \leq C_{2} \| w_{j} \|. \] (3.11)

Form above, we have
\[ \frac{d}{dt} (\| \nabla \nabla u \|^{2} + \| \nabla \nabla w \|^{2}) + 2 \| \nabla \nabla w \|^{2} \leq (C_{2} + 2 C_{2}) \| \nabla \nabla u \|^{2} + (C_{2} + 2 C_{2}) \| w_{j} \|. \] (3.12)

Then
\[ \frac{d}{dt} (\| \nabla \nabla u \|^{2} + \| \nabla \nabla w \|^{2}) \leq (C_{2} + 2 C_{2}) \| \nabla \nabla u \|^{2} + (C_{2} + 2 C_{2}) \| w_{j} \|. \] (3.13)

According to \( \| \nabla \nabla u \|^{2} \geq \varepsilon \| \nabla \nabla w \| \), then
\[ \frac{d}{dt} (\| \nabla \nabla u \|^{2} + \| \nabla \nabla w \|^{2}) \leq \left( \frac{C_{2} + 2 C_{2}}{\varepsilon} \right) \| \nabla \nabla u \|^{2} + (C_{2} + 2 C_{2}) \| w_{j} \|. \] (3.14)

Taking \( \gamma = \max\{ \frac{C_{2} + 2 C_{2}}{\varepsilon}, C_{2} + 2 C_{2} \} \), we have
\[ \frac{d}{dt} (\| w_{j} \|^{2} + \| \nabla \nabla w \|^{2}) \leq \gamma (\| \nabla \nabla u \|^{2} + \| \nabla \nabla w \|^{2} + \| w_{j} \|) \] (3.15)

By using Gronwall inequality, we obtain
\[ \| w_{j} \|^{2} + \| \nabla \nabla w \|^{2} \leq \gamma (\| \nabla \nabla u \|^{2} + \| \nabla \nabla w(0) \|^{2} + \| w_{j}(0) \|) e^{\gamma t}. \] (3.16)

Therefore
\[ u = \nu \]

So we get the uniqueness of the solution.

### 3.2 Global attractor
Theorem 3.1. [11] Let $E_1$ be a Banach space, and $(S(t))(t \geq 0)$ are the semigroup operator on $E_1$.

$S(t): E_1 \rightarrow E_1$, $S(t+s) = S(t)S(s)$ $(\forall t, s \geq 0)$, $S(0) = I$, where $I$ is a unit operator. Set $S(t)$ satisfy the follow conditions.

1) $S(t)$ is uniformly bounded, namely $\forall R > 0$, $\exists u \subseteq \mathbb{R}^2 \leq R$, it exists a constant $C(R)$, so that

$\Box S(t)u \subseteq C(R)$ $(t \in [0, +\infty))$;

2) It exists a bounded absorbing set $B_0 \subset E_1$, namely, $\forall B \subset E_1$, it exists a constant $t_0$, so that $S(t)B \subset B_0$ $(t \geq t_0)$;

Where $B_0$ and $B$ are bounded sets.

3) When $t > 0$, $S(t)$ is a completely continuous operator $A$.

Therefore, the semigroup operator $S(t)$ exists a compact global attractor.

Theorem 3.2. [12] Under the assume of Lemma 1, Lemma 2, Theorem 3.1, equations have global attractor

$A = \omega(B_0) = \bigcap_{t \geq 0} \bigcup_{S(t)B_0} S(t)B_0$.

Where $B_0 = \{(u, v) \in H^{2m}_0(\Omega) \times H^m_0(\Omega) | \Box (u, v) \Box^2_{H^{2m}_0, H^m_0} \leq u^2_{H^{2m}_0} + v^2_{H^m_0} \leq R_0 + R_1\}$. $B_0$ is the bounded absorbing set of $H^{2m}_0(\Omega) \times H^m_0(\Omega)$ and satisfies

(1) $S(t)A = A$, $t > 0$;

(2) $\lim_{t \rightarrow \infty} \text{dist}(S(t)B, A) = 0$, here $B \subset H^{2m}_0 \times H^m_0$ and it is a bounded set,

$\lim_{t \rightarrow \infty} \text{dist}(S(t)B, A) = \sup_{x \in B} (\inf_{y \in A} \|S(t)x - y\|_{H^{2m}_0 \times H^m_0}) \rightarrow 0$, $t \rightarrow \infty$.

Proof. Under the conditions of Theorem 3.1, it exists the solution semigroup $S(t)$, $S(t): H^{2m}_0(\Omega) \times H^m_0(\Omega) \rightarrow H^{2m}_0(\Omega) \times H^m_0(\Omega)$, here $E_1 = H^{2m}_0(\Omega) \times H^m_0(\Omega)$.

(1) from Lemma 2.1 to Lemma 2.2, we can get that $\forall B \subset H^{2m}_0 \times H^m_0$ is a bounded set that includes in the ball

$\Box (u, v) \Box^2_{H^{2m}_0 \times H^m_0} \leq R$.

This shows that $S(t)$ $(t \geq 0)$ is uniformly bounded $H^{2m}_0(\Omega) \times H^m_0(\Omega)$.

(2) Furthermore, for any $(u_0, v_0) \in H^{2m}_0(\Omega) \times H^m_0(\Omega)$, when $t \geq \max\{t_1, t_2\}$, we have,
\[ \Box S(t)(u_{0},v_{0}) \leq \frac{1}{2} \| h_{0} \| _{H}^{2} + \Box v \| _{H}^{2} \leq R_{0} + R_{1} \]

So we get \( B_{0} \) is the bounded absorbing set.

(3) Since \( E_{1} = H^{2} \times H^{m} \rightarrow E_{0} = H^{m} \times L^{2} \) is compact embedded, which means that the bounded set in \( E_{1} \) is the compact set in \( E_{0} \), so the semigroup operator \( S(t) \) exists a compact global attractor \( \mathcal{A} \).

4 The estimates of the upper bounds of Hausdorff and fractal dimensions for the global attractor

We rewrite the problems (1.1)-(1.3):

\[
\begin{align*}
\frac{du}{dt} + A^{2}u + \| A^{-\frac{1}{2}}u \| _{H}^{2}A^{n}u + h(u) &= f(x), \\
U_{0} &= \xi, \quad U_{0}(0) = \zeta.
\end{align*}
\]

(4.1)

Let \( Au = -\Delta u \), where \( \Omega \) is a bounded domain in \( R^{N} \) with smooth boundary \( \partial \Omega \). The linearized equations of the above equation as follows

\[
U_{0} + AU = FU, \quad (4.2)
\]

Let \( U_{0} \in H^{m} \), \( U_{0}(t) \) is the solution of problems (4.20)-(4.21). We can prove that the problems (4.20)-(4.21) have a unique solution \( U \in L_{0}^{2}(0,T), H^{m}(\Omega), U_{0} \in L_{0}^{2}(0,T), L_{0}^{2}(\Omega) \). The equation (4.20) is the linearized equation by the equation (4.17). Define the mapping \( L_{s}(t)_{u_{0}} : L_{s}(t), \xi = U(t), \) here \( u(t) = s(t)u_{0}, \) let \( \varphi_{0} = (u_{0}, u_{1}) \),

\[
\varphi_{0} = \varphi_{0} + (\xi, \quad \zeta) = (u_{0} + \xi, \quad u + \zeta) \text{let :}
\]

\[
\begin{align*}
\| \varphi_{0} \| _{H^{m}} \leq R_{1}, \quad \| \varphi_{0} \| _{L^{2}} \leq R_{2}, \quad E_{0} = v \times H, \quad E_{0} = v \times H, \quad v := H^{m}(\Omega) \text{ } H := L^{2}(\Omega), \quad S(t)\varphi_{0} = \varphi(t) = \{u(t), \quad u_{1}(t)\},
\end{align*}
\]

\[ S(t)\varphi_{0} = \{ \varphi(t), \varphi_{1}(t) \} \]

Lemma 4.19 Assume \( H \) is a Hilbert space, \( E_{0} \) is a compact set of \( H \), \( S(t) : E_{0} \rightarrow H \) is a continuous mapping, satisfy the following conditions.

1) \( S(t)E_{0} = E_{0}, \quad t > 0; \)

2) if \( S(t) \) is Frechet differentiable, it exists a bounded linear differential operator \( L(t,\varphi_{0}) \in C^{*}(R^{*};L(E_{0},E_{0})), \)

\[ \forall t > 0 \text{ that is} \]

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The proof of lemma 4.1 see ref.[9], is omitted here. According to Lemma 4.1, we can get the following theorem:

**Theorem 4.1** Let $A$ is the global attractor that we obtain in section 3. In that case, $A$ has finite Hausdorff dimensions and fractal dimensions in $(H^{2n}(\Omega) \cap H^m(\Omega)) \times H^m(\Omega)$, that is $d_H(A) \leq n, d_f(A) \leq \frac{6n}{5}$.

**Proof.** Firstly, we rewrite the equations (4.17), (4.18) into the first order abstract evolution equations in $E_0$.

Let $\Psi = R_x \varphi = \{ u, u_r, \varepsilon u \}$, let $R_x : \{ u, u_r \} \to \{ u, u_r, \varepsilon u \}$ is an isomorphic mapping. So let is the global attractor of \{S(t)\}, then $R_x A$ is also the global attractor of \{S_x(t)\}.

Let $\Psi(t) = \{ u, u_r, \varepsilon u \}$, then $\Psi$ satisfies as follows:

$$\Psi_t + \Lambda_x \Psi + \bar{h}(\Psi) = f,$$

$$\Psi(0) = \{ u_0, u_1, \varepsilon u_0 \}^T \quad (4.4)$$

Where $\Psi = \{ u, u_r, \varepsilon u \}^T$, $\bar{h}(\Psi) = \{ 0, h(u_r) \}^T$, $f = \{ 0, f(x) \}^T$.

$$\Lambda_x = \begin{pmatrix} \varepsilon I & -1 \\ \frac{\varepsilon}{\varepsilon} & -I \end{pmatrix}$$

$$\Psi_t = F(\Psi) = f - \Lambda_x \Psi - \bar{h}(\Psi) \quad (4.7)$$

$$P_t + \Lambda_x P + \bar{h}(\Psi) = 0 \quad (4.8)$$

Where $P = \{ U, U_r, \varepsilon U \}^T$, $\bar{h}(\Psi) P = \{ 0, h(u_r)U \}^T$ The initial condition (4.3) can be written in the following form:

$$P(0) = w, w = \{ \xi, \zeta \} \in E_0 \quad (4.10)$$

We take $n \in N$, then consider the corresponding $n$ solution: $(P = P_1, P_2, \ldots P_n, P_r \in E_0)$ of the initial values:

$$(w = w_1, w_2, \ldots, w_n) \in E_0$$

in the equations (4.8)-(4.10)

So there is $\int_0^t P_1(t) \Lambda P_2(t) \Lambda \cdots \Lambda P_n(t) d\tau = \int_0^t Tr F_i(S_i(t) \Psi(t) Q(t) d\tau$
Form \( \psi (\tau) = S_x(\tau)\Psi_o \), we get \( S_x(\tau) : [u_0, v_i = u_i + \varepsilon u_0] \rightarrow \{ u(\tau), v(\tau) = u_i(\tau) + \varepsilon u(\tau) \} \),

\( \psi (\tau) = \{ u(\tau), v_i(\tau) = u_i(\tau) + \varepsilon u(\tau) \} \), here \( u \) is the solution of problems (4.1); \( \Lambda \) represents the outer product,

\( Tr \) represents the trace, \( Q_x(\tau) = Q_x(\tau, \Psi_o; w_1, w_2, \cdots w_n) \) is an orthogonal projection from the space \( E_u = u \times H \) to the subspace spanned by \( \{ P_1(\tau), P_2(\tau), \cdots P_n(\tau) \} \). For a given time \( \tau \), let \( \phi_j(\tau) = \{ \xi_j(\tau), \zeta_j(\tau) \} \), \( j = 1, 2, \cdots n \).

\( \{ \phi_j(\tau) \}_{j=1}^{n} \) is the standard orthogonal basis of the space \( Q_x(\tau)_{E_u} = \text{span} \{ P_1(\tau), P_2(\tau), \cdots P_n(\tau) \} \).

From the above, we have

\[
\text{Tr} F_i(\Psi(\tau)) \cdot Q_x(\tau) = \sum_{j=1}^{n} F_i(\Psi(\tau)) \cdot Q_x(\tau) \phi_j(\tau) \phi_j(\tau)_{E_u}
\]

\[
= \sum_{j=1}^{n} F_i(\Psi(\tau)) \phi_j(\tau) \phi_j(\tau)_{E_u} \quad (4.11)
\]

Where \((\cdot, \cdot)_{E_u}\) is the inner product in \( E_u \). Then \( \{ (\xi, \zeta), (\xi, \zeta) \}_{E_u} = (\xi, \zeta) + (\zeta, \xi) ;

\[
(F_i(\Psi) \phi_j, \phi_j)_{E_u} = -(\lambda \phi_j, \phi_j)_{E_u} - (h_i(u, \xi_j, \zeta_j) ;
\]

\[
(\lambda \phi_j, \phi_j)_{E_u} = \varepsilon \| \xi \|^2 + (\| A^2 u \|^2 - \varepsilon A^2 + \varepsilon A^2) (\zeta_j, \xi_j)
\]

\[
- (\xi_j, \zeta_j) + (A^m - \varepsilon I) (\zeta_j, \zeta_j) \quad (4.12)
\]

\[
\geq a \left( \| \xi \|^2 + \| \zeta \|^2 \right)
\]

Where \( a := \min \left[ \frac{2 \varepsilon + \varepsilon^2}{2}, \frac{2 \varepsilon + \varepsilon^2}{2} \right] \).

Now, suppose that \( \{ u_0, u_1 \} \in A \), according to Theorem 3.3, \( A \) is a bounded absorbing set in \( E_i \);

\[
\Psi(t) = (u(t), u_i(t) + \varepsilon u(t)) \in D(A); D(A) = \{ u \in v, Au \in H \}.
\]

Then there is a \( s \in [0, 1] \) to make the mapping \( h_i : D(A) \rightarrow \rho(v_s, H) \). At the same time, there are the following results:
\[ R_A = \sup_{i \leq \xi, \zeta \in A} |A\xi| < \infty \]

\[ \sup_{a \in D(\xi)} h_{i}(u_{j}) \leq r < \infty \quad (4.13) \]

Where \( h_{i}(u_{j}) \xi, \zeta \) meets: \( h_{i}(u_{j}) \xi, \zeta \leq r \| \xi \| \| \zeta \| \). Comprehensive above can be obtained

\[ (F_{i}(\Psi) \phi_{j}, \phi_{j})_{E_{n}} \leq -a(\| \xi \| ^{2} + \| \zeta \| ^{2}) + r \| \xi \| \| \zeta \| \]

\[ \leq -\frac{a}{2}(\| \xi \| ^{2} + \| \zeta \| ^{2}) + \frac{r^{2}}{2a} \| \xi \| \| \zeta \| \quad (4.14) \]

Where \( \| \xi \| ^{2} + \| \zeta \| ^{2} = \| \phi_{s} \| _{E_{n}} = 1 \), due to \( \{ \phi_{j}(\tau) \}_{j=1}^{n} \) is a standard orthogonal basis in \( Q_{s}(\tau)_{E_{n}} \)

So

\[ \sum_{j=1}^{n} F_{i}(\Psi(\tau)) \phi_{j}(\tau), \phi_{j}(\tau)_{E_{n}} \leq -\frac{na}{2} + \frac{r^{2}}{2a} \| \xi \| _{E_{n}} \quad (4.15) \]

Almost to all \( \tau \), making

\[ \| F_{i}(\Psi(\tau)) \| _{K} \leq \sum_{j=1}^{n} \lambda_{j}^{-1} \quad (4.16) \]

So

\[ TrF_{i}(\Psi(\tau)) \cdot Q_{s}(\tau) \leq -\frac{na}{2} + \frac{r^{2}}{2a} \sum_{j=1}^{n} \lambda_{j}^{-1} \quad (4.17) \]

Let us assume that \( \{ u_{0}, u_{j} \} \in A \), is equivalent to \( \Psi_{0} = \{ u_{0}, u_{1} + \epsilon u_{0} \} \in R_{A} \)

Then

\[ q_{n}(t) = \sup_{\Psi_{0} \in R_{A}} \sup_{u \in E_{s}} (\int_{0}^{t} TrF_{i}(S_{j}(\tau) \Psi_{0}) \cdot Q_{s}(\tau) d \tau) \cdot j = 1,2, \ldots, n. \]

\[ q_{n} = \lim_{t \to \infty} \sup_{t} q_{n}(t) \quad (4.18) \]

According to (4.17), (4.18), so

\[ q_{n}(t) \leq -\frac{na}{2} + \frac{r}{2a} \sum_{j=1}^{n-1} \lambda_{j}^{-1} \]
\[ q_n \leq -\frac{na}{2} + \frac{r}{2a} \sum_{j=1}^{n-1} \lambda_j^{r-1} \]  

(4.19)

Therefore, the Lyapunov exponent of \( A \) (or \( R_x \)) is uniformly bounded

\[ \mu_1 + \mu_2 + \cdots + \mu_n \leq -\frac{na}{2} + \frac{r^2}{2a} \sum_{j=1}^{n} \lambda_j^{r-1} \]  

(4.20)

From what has been discussed above, it exists \( n > 1 \) and \( s \in [0,1] \), \( a, r \) are constants, then

\[ \frac{1}{n} \sum_{j=1}^{n} \lambda_j^{r-1} \leq \frac{a^2}{6r^2} \]  

(4.21)

\[ q_n \leq -\frac{na}{2} (1 - \frac{r^2}{a} \sum_{j=1}^{n} \lambda_j^{r-1}) \leq -\frac{5na}{12} \]  

(4.22)

\[ (q_j)_+ \leq -\frac{r^2}{2a} \sum_{i=1}^{j} \lambda_i^{r-1} \leq \frac{na}{12}, j = 1, 2 \cdots n. \]  

(4.23)

So, we immediately to the Hausdorff dimension and fractal dimension are respectively

\[ d_H (A) < n , \ d_F (A) < \frac{4}{5} n . \]

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6 References


