Convergence Theorems of Iterative Schemes For Nonexpansive Mappings

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ABSTRACT

In this paper, we give a type of iterative scheme for a sequence of nonexpansive mappings and we study the strongly convergence of these schemes in real Hilbert space to a common fixed point which is also a solution of a variational inequality. Also there are some consequences of this result in convex analysis.

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Functional Analysis

INTRODUCTION AND PRELIMINARIES

Let X be a Hilbert space, \( \emptyset \neq C \) be a convex closed subset of X and A be a multi-valued mapping with domain \( D(A) \) and range \( R(A) \). The mapping A is called monotone mapping if the following inequality holds

\[
< x_1 - x_2, y_1 - y_1 \geq 0, \forall x_1 \in D(A), \ \forall y_1 \in R(A).
\]

Also, any mapping A is called maximal monotone mapping of A if the graph \( \mathcal{G}(T) \) of T is not properly contained in the graph of any other monotone mapping, where

\[
\mathcal{G}(T) = \{(u, v) \in X \times X; u \in A(x)\}.
\]

Monotone mappings play a crucial role in modern nonlinear analysis and optimization, see the books [1,2,3,4,5].

The single valued nonexpansive itself-mapping on \( C \) is defined as: \( J_{r_n} = (I + r_nA)^{-1}(x) \), and is called the resolvent mapping on \( C \), where \( < r_n > \) be a sequence of positive real numbers. In [6], Moudafi, studied the strong convergence of both the following iterative schemes in Hilbert space

\[
x_i = tf(x_i) + (1 - t)T_{x_i}, \quad \text{as } t \to \infty \quad (1)
\]

\[
x_{n+1} = \alpha_n f(x_n) + (1 - t)T_{x_n}, \quad \text{as } n \to \infty \quad (2)
\]

where \( f \) be a contraction mapping, \( T \) is a nonexpansive mapping and \( < \alpha_n > \) be a sequence in \( (0,1) \). In this paper we study the strongly convergence of common fixed point of the sequence of nonexpansive mapping which is also a solution of variational inequality,

\[
< (I - f)\delta, x - \delta > \leq 0, \quad x \in A^{-1}(0)
\]

Now, we recall some definitions and lemmas which we used in the proofs:

**Definition 1.** [6] and [7]

1. A mapping \( T : C \to X \) is called a Lipschitz continuous with constant \( \alpha > 0 \)

\[
||Tx - Ty|| \leq \alpha ||x - y||, \ \text{for any } x, y \in C
\]

2. If \( \alpha \in (0,1) \) \( \Rightarrow T \) is called a contraction mapping.

3. If \( \alpha = 1 \) \( \Rightarrow T \) is called a nonexpansive mapping.

**Definition 2.** [6] and [7] A mapping \( T : C \to X \) is called

1. firmly nonexpansive mapping if for any \( x, y \in C \) then,

\[
||Tx - Ty||^2 + ||(I - T)x - (I - T)y||^2 \leq ||x - y||^2
\]

2. strongly nonexpansive mapping if it is nonexpansive and for any \( < x_n > \) and \( < y_n > \) are sequences in \( C \) such that \( < x_n - y_n > \) is bounded and \( ||x_n - y_n|| \to 0 \) it follows that \( (x_n - y_n) - (Tx_n - Ty_n) \to 0 \).
Note that Both firmly nonexpansive and strongly nonexpansive imply nonexpansive.

**Theorem 3.** [7] If $T$ be a mapping from $X$ into $X$, then the following are equivalent

1. $T$ is firmly nonexpansive
2. $(I - T)$ is firmly nonexpansive
3. $(2I - T)$ is nonexpansive
4. $\|Tx - Ty\| \leq <x - y, Tx - Ty>$ for all $x, y \in X$
5. $0 \leq <x - y, Tx - Ty>$ for all $x, y \in X$

**Lemma 4.** [8] If $X$ be a real Hilbert space, $\emptyset \neq C$ be a convex closed in $X$ and $T$ be an nonexpansive mapping with $F(T) \neq \emptyset$, suppose that $<x_n>$ converge weakly to $x$, $\lim (1 - T)x_n = y$, then $(1 - T)x = y$.

**Lemma 5.** [9] If $\{x_n\}$ be a sequence of nonnegative real number such that $a_n < \{x_n\}; n \geq 0, a_{n+1} \leq 1 - y_n a_n + y_n S_n$ where $<S_n>$ be a sequence in the real number and $<y_n>$ be a sequence in $(0,1)$ such that $\sum |S_n| < \infty$ and $0 \leq \lim n \to \infty \sup a_n / y_n$. Then $a_n \to 0$ as $n \to \infty$.

**Lemma 6.** [10] Let $\emptyset \neq C$ convex closed in $X$ and $T$ be a multivalued nonexpansive mapping. If $x_n$ convergence weakly to $p$ and $\lim \|x_n - Tn(p)\| = 0$, then $p \in F(T)$.

**2. MAIN RESULTS** Let $X$ be a real Hilbert space and $C$ be a nonempty convex closed subset of $X$. Denote by:

- $F$ is the class of the sequence $<f_n>$ of mappings on $C$ such that
  \[ \|f_n(x_n) - f_n(x_{n-1})\| \leq \|f_{n-1}(x_n) - f_{n-1}(x_{n-1})\| \]
- $T_i$ be a mapping on $C$ such that $T_i(x) = f_n(x) + (1-t)f_n(x); t > 0$

Now, we give the following definition.

**Definition 2.1.** Let $<T_n>$ be a sequence of mappings on $C$, then $p \in C$ is called asymptotic common fixed point of $<T_n>$ if there exist a sequence $<x_n>$ in $C$ converges weakly to $p$ and $\lim n \to \infty \|x_n - Tn(x_n)\| = 0$.

In this paper, we study the strong convergence of types of iterative schemes in real Hilbert space.

**Remark 2.2.** If $<f_n>$ be a sequence of nonexpansive mappings then $T_i$ is also nonexpansive.

**Proof** For all $x, y \in C$,

$\|T_i(x) - T_i(y)\| \leq t\|f_n(x) - f_n(y)\| + (1 - t)\|f_n(x) - f_n(y)\|$

$\leq t\|x - y\| + (1 - t)\|x - y\|$

$\leq \|x - y\|$

**Theorem 2.3.** Let $A$ be a maximal multivalued mapping, $<f_n>$ be a sequence of bounded and contraction mappings on $C$ and $A^{-1}(0) = \emptyset$. Then $<x_n>$ converges strongly to the point $\bar{x}$, where $\bar{x} = \mathcal{P}_E(f_n(\bar{x}))$ or $\bar{x}$ is the unique solution of variation of variational inequality $<1 - f_n(\bar{x}), x - \bar{x}> \geq 0$, $x \in E = A^{-1}(0)$.

**Proof** Let $p \in A^{-1}(0)$

$\|x_t - p\| \leq t\|f_n(x_t) - p\| + (1 - t)\|f_n(x_t) - p\|$

$\leq t\|f_n(x_t) - p\| + (1 - t)\|x_t - p\|$

$t\|x_t - p\| \leq t\|f_n(x_t) - p\|$

$\|x_t - p\| \leq \|f_n(x_t) - f_n(p)\| + \|f_n(p) - p\|$

$\leq \alpha\|x_t - p\| + \|f_n(p) - p\| ; \alpha = \max\{\alpha_0, \alpha I\}; 0 \leq \alpha < 1$

$\|x_t - p\| \leq \frac{1}{1 - \alpha}\|f_n(p) - p\|$

But $<f_n>$ is bounded sequence, and hence $<x_n>$ is bounded sequence, So $<x_n>$ also bounded.

$\|x_t - f_n(x_t)\| = t\|f_n(x_t) - f_n(x_t)\| = 0$ as $t \to 0$

Since $<x_n>$ is bounded then there exists a subsequence $<x_n>$ of $<x_t>$ such that $x_n \to \bar{x}$.

By lemma (1.4), we get $\bar{x} \in A^{-1}(0)$
Now, since $x_{t} - \tilde{x} = t(f_{t}(x_{t}) - \tilde{x}) + (1 - t)(I_{m}(x_{t}) - \tilde{x}),$

$\|x_{t} - \tilde{x}\|^{2} = t < f_{t}(x_{t}) - \tilde{x}, x_{t} - \tilde{x}> + (1 - t) < I_{m}(x_{t}) - \tilde{x}, x_{t} - \tilde{x}>$

$\leq t < f_{t}(x_{t}) - \tilde{x}, x_{t} - \tilde{x}> + \|x_{t} - \tilde{x}\|^{2}$

$\|x_{t} - \tilde{x}\|^{2} \leq < f_{t}(x_{t}) - \tilde{x}, x_{t} - \tilde{x}>$

$\leq \alpha \|x_{t} - \tilde{x}\|^{2} + < f_{t}(\tilde{x}) - \tilde{x}, x_{t} - \tilde{x}>$

$\alpha = \sup \{\alpha_{t} : t \in N\} \text{such that } 0 < \alpha < 1$

$\|x_{t} - \tilde{x}\|^{2} \leq \frac{1}{1 - \alpha} < f_{t}(\tilde{x}) - \tilde{x}, x_{t} - \tilde{x}>$

And hence, $\|x_{m} - \tilde{x}\|^{2} \leq \frac{1}{1 - \alpha} < f_{n}(\tilde{x}) - \tilde{x}, x_{m} - \tilde{x}>$

But $x_{m} \to \tilde{x}$, then as $n \to \infty$ we get

$< f_{n}(\tilde{x}) - \tilde{x}, x_{m} - \tilde{x}> \to 0$ and hence, $\|x_{t} - \tilde{x}\| \to 0$

Now, to prove that $\tilde{x}$ is unique solves of the variational inequality.

Since, $x_{t} = t_{n} f_{t}(x_{t}) + (1 - t_{n}) I_{m} x_{t} \Rightarrow (1 - f_{n}(x_{t})) = -\left(\frac{1}{1 - t_{n}}\right) (1 - I_{m})(x_{t})$

And for all $z \in A^{-1}(0)$

$< (1 - f_{n}(x_{t})), x_{t} - z > \geq \frac{1}{1 - t_{n}} < (1 - I_{m})(x_{t}), x_{t} - z >$

$= -\frac{1}{1 - t_{n}} < (1 - I_{m})(x_{t}) - (1 - I_{m})(\tilde{x}), x_{t} - \tilde{x} >$

$\leq 0 \text{ as } (1 - I_{m}) \text{ is monotone.}$

Therefore, $\tilde{x}$ is a solution of variational inequality

$< (1 - f_{n})(x_{t}), x_{t} - z > \leq 0 \text{, } \forall \text{ } z \in A^{-1}(0)$

To prove the uniqueness, suppose that

$x_{m} \to \tilde{x} \in E = A^{-1}(0)$ and $\tilde{x}$ is solution of variational inequality

$< (1 - f_{n})(\tilde{x}), \tilde{x} - \tilde{x}> \leq 0 \text{, } (3)$

Interchange $\tilde{x}$ and $\tilde{x}$

$< (1 - f_{n})(\tilde{x}), \tilde{x} - \tilde{x}> \leq 0 \text{, } (4)$

Adding up (3) and (4) we have

$< \tilde{x} - \tilde{x}, (1 - f_{n})(\tilde{x}) - (1 - f_{n})(\tilde{x}) > \leq 0$

By lemma (1.5), we get $\tilde{x} = \tilde{x}$

**corollary 2.4.** Let $A$ be a maximal multivalued mapping and $< T_{n} >$ be a sequence of firmly non expansive. If the scheme $< x_{n} >$ is defined as:

$x_{n+1} = \alpha_{n} f_{n}(x_{n}) + \beta_{n} T_{n}^{\alpha_{n}}(x_{n}) + (1 - \gamma_{n})(x_{n})$

Where $< \alpha_{n} >, < \gamma_{n} >, < \gamma_{n} > and < \beta_{n} >$ as in theorem (2.3) and

$T_{n}^{\alpha_{n}}(x_{n}) = (1 - \alpha_{n}) x_{n} + \alpha_{n} T_{n}(x_{n})$

Then $< x_{n} >$ converges strongly to an asymptotic common fixed point of $T_{n}^{\alpha_{n}}, \forall n \in N.$

**Proof** For any $x, y \in X$

$\|T_{n}^{\alpha_{n}}(x) - T_{n}^{\alpha_{n}}(y)\| \leq (1 - \alpha_{n}) \|x - y\| + \alpha_{n} \|T_{n}(x) - T_{n}(y)\|$

$\leq (1 - \alpha_{n}) \|x - y\| + \alpha_{n} \|x - y\|$

$= \|x - y\|$

Therefore, $< T_{n}^{\alpha_{n}} >$ is a sequence of nonexpansive. Then by theorem (2.3) we get the result.
Theorem 2.5. Let A be a maximal monotone multivalued mapping, \(< f_n >\) be a sequence of contraction mapping on C and \(< T_n >\) be a sequence of nonexpansive mapping on C, \(< f_n >\) and \(< T_n >\) lines in \(\mathcal{F}\) such that \(A\) is nonempty. If the iterative scheme \(< x_n >\) is defined as:

\[ x_{n+1} = \alpha_n f_n(x_n) + \beta_n T_n(x_n) + (1 - \gamma_n)j_n(x_n) \]

Where \(\alpha_n > 0\) and \(\beta_n > 0\) are decreasing sequences in \([0,1]\) converges to 0, such that

\[ \sum_{n=0}^{\infty} \alpha_n = \infty \quad \text{and} \quad \alpha_n + \beta_n + (1 - \gamma_n) = 1. \]

Then the iterative scheme \(< x_n >\) converges strongly to an asymptotic common fixed point of \(T_n, \forall \ n \in N\).

Proof. Let \(p \in A^{-1}(0) \cap (\bigcap \{ F(T_n) \}) \cap (\bigcap \{ F(f_n) \})\)

\[ \| x_{n+1} - p \| \leq \alpha_n \| f_n(x_n) - p \| + \beta_n \| T_n(x_n) - p \| + (1 - \gamma_n) \| j_n(x_n) - p \| \]

\[ \leq \alpha_n \| x_n - p \| + \beta_n \| x_n - p \| + (1 - \gamma_n) \| x_n - p \| \]

where \(\alpha = \sup(\alpha_i, i \in N)\) and \(0 < \alpha < 1\)

\[ \| x_{n+1} - p \| \leq (\alpha_n + \beta_n + (1 - \gamma_n)) \| x_n - p \| \]

\[ \| x_{n+1} - p \| \leq \| x_n - p \| \Rightarrow \| x_n \| \text{ is bounded sequence, So } \| f_n \|, \| T_n \| \text{ and } \| j_n \| \text{ are bounded.} \]

Now, since \(\| f_n \| > 0, \| T_n \| > 0\) lies in \(\mathcal{F}\). Therefore,

\[ \| x_{n+1} - x_n \| \leq \alpha_{n-1} \| f_n(x_n) - f_{n-1}(x_{n-1}) \| + \beta_{n-1} \| T_n(x_n) - T_{n-1}(x_{n-1}) \| + (1 - (\alpha_{n-1} + \beta_{n-1})) \| j_n(x_n) - j_{n-1}(x_{n-1}) \| \]

\[ \leq \alpha_{n-1} \| f_n(x_n) - f_{n-1}(x_{n-1}) \| + \beta_{n-1} \| T_n(x_n) - T_{n-1}(x_{n-1}) \| + \alpha_{n-1} \| x_n - x_{n-1} \| + (1 - (\alpha_{n-1} + \beta_{n-1})) \| j_n(x_n) - j_{n-1}(x_{n-1}) \| \]

And hence, \(\| x_{n+1} - x_n \| \to 0\) as \(n \to \infty\) \(\quad (5)\)

\[ \| x_n - T_n x_n \| \leq \| x_n - x_{n+1} \| + \| x_{n+1} - T_n x_n \| \]

\[ < \| x_{n+1} - x_n \| + \alpha_n \| f_n(x_n) \| + 2\beta_n \| f_n(x_n) \| \]

\[ + (\alpha_n + \beta_n) \| j_n(x_n) \| \]

But \(\| f_n \| > 0\) and \(\| j_n \| > 0\) are bounded and by (5), we get

\[ \| x_n - T_n x_n \| \to 0 \quad \text{as} \quad n \to \infty \quad (6) \]

Since \(\| x_n \|\) is bounded sequence then there exists as a subsequence \(< x_{n_k} >\) of \(< x_n >\) such that \(x_{n_k} \to \bar{x}\).

By equation (6) and by using lemma (1.6), we get \(\bar{x} \in \bigcap \{ F(T_n) \}\)

\[ \| x_{n-1} - \bar{x} \| \leq \alpha_n \| f_n(x_n) - \bar{x} \| + \beta_n \| T_n(x_n) - \bar{x} \| + (1 - \gamma_n) \| j_n(x_n) - \bar{x} \| \]

\[ \leq \alpha_n \| f_n(x_n) - \bar{x} \| + (1 - \gamma_n) \| f_n(x_n) - \bar{x} \| + (\alpha_n + \beta_n) \| j_n(x_n) - \bar{x} \| \]

\[ = (1 - \alpha_n) \| x_n - \bar{x} \| + \alpha_n \| f_n(x_n) - \bar{x} \| + (\alpha_n + \beta_n) \| j_n(x_n) - \bar{x} \| \]

By lemma (1.5), we get \(\| x_n - \bar{x} \| \to 0\) as \(n \to \infty\). And hence \(< x_n >\) converges strongly to an asymptotic fixed point of \(T_n, \forall n \in N\).
Corollary 2.6. Let $A$ be a maximal monotone multivalued mapping, $f$ be a contraction self-mapping on $C$ and $T$ be a non-expansive self-mapping on $C$ such that $A^{-1}(0) \cap (F(T) \cap (F(T)') \neq \emptyset$ and $f$ and $T$ lines in $F$. If the iterative scheme $x_{n+1}$ is defined as:

$$x_{n+1} = \alpha_n f(x_n) + \beta_n T(x_n) + (1 - \gamma_n)T_r(x_n)$$

Where $\alpha_n > 0$ and $\beta_n > 0$ are decreasing sequences in $(0,1)$ converges to 0, such that

1. $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\alpha_n + \beta_n + (1 - \gamma_n) = 1$.

2. $\frac{1}{2} \leq \alpha_n + \beta_n < 1$ and $\sum_{n=0}^{\infty} \|f(x_n)\| + \|T_r(x_n)\| < \infty$. Then the iterative scheme $x_{n+1}$ converges strongly to an asymptotic common fixed point of $T_r$, $\forall n \in \mathbb{N}$.

Corollary 2.7. Let $A$ be a maximal monotone multivalued mapping, $f$ be a contraction self-mapping on $C$ and $T$ be a non-expansive mapping on $C$ such that $A^{-1}(0) \cap (\cap F(T) \cap (\cap F(T)) \neq \emptyset$ and $f$ and $T$ lines in $F$. If the scheme $x_{n+1}$ is defined as:

$$x_{n+1} = \alpha_n f(x_n) + \beta_n T(x_n) + (1 - \gamma_n)T_r(x_n)$$

Where $\alpha_n > 0$ and $\beta_n > 0$ are decreasing sequences in $(0,1)$ converges to 0, such that

1. $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\alpha_n + \beta_n + (1 - \gamma_n) = 1$.

2. $\frac{1}{2} \leq \alpha_n + \beta_n < 1$ and $\sum_{n=0}^{\infty} \|f(x_n)\| + \|T_r(x_n)\| < \infty$. Then the iterative scheme $x_{n+1}$ converges strongly to an asymptotic fixed point of $T_r$, $\forall n \in \mathbb{N}$.

Corollary 2.8. Let $A$ be a maximal monotone multivalued mapping, $f$ be a sequence of contraction mapping on $C$ and $T$ be a sequence of non-expansive mapping on $C$ such that $A^{-1}(0) \cap (\cap F(T) \cap (\cap F(T)) \neq \emptyset$ and $f$ and $T$ lines in $F$. If the scheme $x_{n+1}$ is defined as:

$$x_{n+1} = \alpha_n f(x_n) + \beta_n T(x_n) + (1 - \gamma_n)T_r(x_n)$$

Where $\alpha_n > 0$ and $\beta_n > 0$ are decreasing sequences in $(0,1)$ converges to 0, such that

1. $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\alpha_n + \beta_n + (1 - \gamma_n) = 1$.

2. $\frac{1}{2} \leq \alpha_n + \beta_n < 1$ and $\sum_{n=0}^{\infty} \|f(x_n)\| + \|T_r(x_n)\| < \infty$. Then the iterative scheme $x_{n+1}$ converges strongly to an asymptotic fixed point of $T_r$, $\forall n \in \mathbb{N}$.

Corollary 2.9. Let $A$ be a maximal multivalued mapping and $T_n$ be a sequence of nonexpansive. If the iterative scheme $x_{n+1}$ is defined as:

$$x_{n+1} = \alpha_n f(x_n) + \beta_n T_n^{\alpha_n}(x_n) + (1 - \gamma_n)T_r(x_n)$$

Where $\alpha_n > 0$, $\beta_n > 0$ as in theorem (2.5) and $T_n^{\alpha_n}(x_n) = (1 - \alpha_n)x_n + \alpha_n T_n(x_n)$

Then $x_{n+1}$ converges strongly to common asymptotic fixed point of $T_n^{\alpha_n}$, $\forall n \in \mathbb{N}$.

Proof. For any $x, y \in X$

$$\|T_n^{\alpha_n}(x) - T_n^{\alpha_n}(y)\| \leq (1 - \alpha_n)\|x - y\| + \alpha_n\|T_n(x) - T_n(y)\|$$

$$\leq (1 - \alpha_n)\|x - y\| + \alpha_n\|x - y\| = \|x - y\|$$

Therefore, $T_n^{\alpha_n}$ is a sequence of nonexpansive. Then by theorem (2.5) we get the result.

Corollary 2.10. Let $A$ be a maximal multivalued mapping and $T_n$ be a sequence of strongly nonexpansive. If the scheme $x_{n+1}$ is defined as:

$$x_{n+1} = \alpha_n f(x_n) + \beta_n T_n^{\alpha_n}(x_n) + (1 - \gamma_n)T_r(x_n)$$

Where $\alpha_n > 0$, $\beta_n > 0$ as in theorem (2.5) and $T_n^{\alpha_n}(x_n) = (1 - \alpha_n)x_n + \alpha_n T_n(x_n)$

Then $x_{n+1}$ converges strongly to common asymptotic fixed point of $T_n^{\alpha_n}$, $\forall n \in \mathbb{N}$.

Corollary 2.11. Let $A$ be a maximal multivalued mapping and $T_n$ be a sequence of firmly nonexpansive. If the iterative scheme $x_{n+1}$ is defined as:

$$x_{n+1} = \alpha_n f(x_n) + \beta_n T_n^{\alpha_n}(x_n) + (1 - \gamma_n)T_r(x_n)$$
Where \( < f_n >, < a_n > \) and \( < b_n > \) as in theorem (2.5) and
\[
T_n^\alpha(x_n) = (1 - a_n) x_n + a_n T(x_n)
\]
Then \( < x_n > \) converges strongly to common asymptotic fixedpoint of \( T_n^\alpha, \forall n \in N \).

**Corollary 2.12.** Let \( A \) be a maximal multivalued mapping and \( T:C \to C \) be a nonexpansive mapping. If the iterative scheme \( < x_n > \) is defined as:
\[
x_{n+1} = \alpha_n f(x_n) + \beta_n T^\alpha(x_n) + (1 - \gamma_n) J_n(x_n)
\]
Where \( < f_n >, < a_n > \) and \( < b_n > \) as in theorem (2.5) and
\[
T^\alpha(x_n) = (1 - a_n) x_n + a_n T(x_n)
\]
Then \( < x_n > \) converges strongly to asymptotic fixed point of \( T^\alpha, \forall n \in N \).

**Corollary 2.13.** Let \( A \) be a maximal multivalued mapping and \( T:C \to C \) be a strongly nonexpansive. If the iterative scheme \( < x_n > \) is defined as:
\[
x_{n+1} = \alpha_n f(x_n) + \beta_n T^\alpha(x_n) + (1 - \gamma_n) J_n(x_n)
\]
Where \( < f_n >, < a_n > \) and \( < b_n > \) as in theorem (2.5) and
\[
T^\alpha(x_n) = (1 - a_n) x_n + a_n T(x_n)
\]
Then \( < x_n > \) converges strongly to asymptotic fixed point of \( T^\alpha, \forall n \in N \).

**Corollary 2.14.** Let \( A \) be a maximal multivalued mapping and \( T:C \to C \) be a firmly nonexpansive. If the iterative scheme \( < x_n > \) is defined as:
\[
x_{n+1} = \alpha_n f(x_n) + \beta_n T^\alpha(x_n) + (1 - \gamma_n) J_n(x_n)
\]
Where \( < f_n >, < a_n > \) and \( < b_n > \) as in theorem (2.5) and
\[
T^\alpha(x_n) = (1 - a_n) x_n + a_n T(x_n)
\]
Then \( < x_n > \) converges strongly to asymptotic fixed point of \( T^\alpha, \forall n \in N \).

### 3. APPLICATIONS

Let \( X \) be a real Hilbert space and \( C \) be a nonempty closed convex of \( X \). If \( f \) be a proper lower semi continuous convex mapping of \( X \) into \( (-\infty, \infty] \) then the sub differential \( \partial f \) of \( f \) is:
\[
\partial f(x) = \{ z \in X : f(y) \geq f(x) + (z, y - x), \forall y \in X \}, \forall x \in X.
\]
Rockefeller [11] proved that \( \partial f \) is maximal monotone multivalued mapping. We recall the normal cone \( N_C(x) \) of \( C \) at \( x \) is define as:
\[
N_C(x) = \{ z \in X : (z, y - x) \leq 0, \forall y \in C \}
\]
And the indicator mapping of \( C \) is define as:
\[
i_C : X \to (-\infty, \infty] \quad \text{such that} \quad i_C(x) = \begin{cases} 0 & \text{if } x \in C \\ \infty & \text{if } x \in C \end{cases}
\]
i_\( C \) is proper lower semicontinuous convex mapping, \( \partial i_C \) is maximal monotone and \( \partial i_C(x) = N_C(x) \). Now, we introduced application for the results presented in this paper.

**Corollary 3.1.** If \( f \) be a proper lower semicontinuous convex mapping of \( X \) into \( (-\infty, \infty] \), \( < f_n > \) be a sequence of bounded and contraction mappings on \( C \) and \( (\partial f)^{-1} \neq \emptyset \). Then \( < x_n > \) converges strongly to the point \( x \), where \( x = p_{E}(f(x)) \) or \( x \) is the unique solution of variation of variational inequality.
\[
< (1 - f_n) x_n - x > \geq 0, \quad x \in E = (\partial f)^{-1}.
\]

**Corollary 3.2.** If \( f \) be a proper lower semicontinuous convex mapping of \( X \) into \( (-\infty, \infty] \), \( < f_n > \) be a sequence of contraction mapping on \( C \) and \( < T_n > \) be a sequence of firmly nonexpansive mapping on \( C \) such that \( (\partial f)^{-1} \cap (\cap F(f_n) \cap (\cap F(T_n)) \neq \emptyset \).
\[
< f_n > \text{ and } < T_n > \text{ lines in } F \text{. If the scheme } < x_n > \text{ is defined as:}
\]
\[
x_{n+1} = \alpha_n f(x_n) + \beta_n T_n(x_n) + (1 - \gamma_n) J_n(x_n)
\]
Where \( < \alpha_n > \) and \( < \beta_n > \) are decreasing sequences in \( (0,1) \) converges to 0, such that
\[
1. \quad \sum_{n=0}^{\infty} \alpha_n = \infty \quad \text{and} \quad \alpha_n + \beta_n + (1 - \gamma_n) = 1.
\]
2. \( \frac{1}{2} \leq \alpha_n + \beta_n < 1 \) and \( \sum_{n=0}^{\infty} \| f_n(x_n) \| + \| J_n(x) \| < \infty \). Then the iterative scheme \( \langle x_n \rangle \) converges strongly to an asymptotic common fixed point of \( T_n \), \( \forall n \in \mathbb{N} \). Then the iterative scheme \( \langle x_n \rangle \) converges strongly to a common asymptotic fixed point of \( T_n \), \( \forall n \in \mathbb{N} \).

REFERENCES