

**DOI:** <https://doi.org/10.24297/ijrem.v10i0.8090>

## **Teaching-learning about optimization problems: a close up through the use of GeoGebra software**

Armando Morales Carballo<sup>1</sup>, Efrén Marmolejo Vega<sup>1</sup>, Edgardo Locia Espinoza<sup>1</sup>, Alejo López González<sup>1</sup>, Angie Damián Mojica<sup>1</sup>

<sup>1</sup>Universidad Autónoma de Guerrero, México

armando280@hotmail.com, afrenmarmolejo@yahoo.com, lociae999@hotmail.com,  
alopezgon@yahoo.com.mx, big\_angie@hotmail.com

### **Abstract**

A teaching-learning proposal is described for the treatment of optimization problems in the pre-university student. The theoretical-methodological elements fall into the heuristic principles and resources and the use of GeoGebra software, on this basis is conceived the identification, selection and formalization of the way of solution, control and assessment as fundamental stages. This work contributes to a proposal to approach the treatment of the study of the variation of the functions, prior to the classic presentation of the mathematical content using the derivative.

**Keywords:** Optimization, relative maximum, heuristic resource, problem, teaching-learning.

**Subject Classification:** Mathematics educative

**Type (Approach):** Cuasi-experimental

### **1. Introduction**

The problem of teaching and learning mathematics, in particular calculus, has been studied from different theoretical and methodological frameworks. Research on this topic show that students have problems with understanding, constructing and interpreting the basics concepts of calculus, such as the concept of function, increasing and decreasing function, maximum and minimum, among others (Pineda, 2013; Delgado, 2013; Ruiz, Gutiérrez, & Garay, 2018; Caves & Delgado, 2016; Valencia & Valenzuela, 2017). These identified problems on teaching and learning of calculus in high school and college have motivated the development of research with the aim of influencing proposals that contribute to the improvement of these situations. Problems with teaching and learning of calculus may be due to different reasons, within them, we can find mathematical and didactic knowledge of the professors, the influence that textbooks and other professors exert in relation to the organization of their class, the characteristics of the students in terms of levels of knowledge development, among others.

With the purpose of influencing the solution to the problem raised, it is described a teaching-learning proposal for the treatment of the problems of optimization in the pre-university student. The theoretical and methodological elements of the proposal fall into the heuristic principles and resources, the GeoGebra software and the basis of methodologies for problem solving a proposal is conceived with the following structure: identification of the problem, selection and formalization of the way of solution, control and assessment, with this work we contribute with a proposal of approaching the treatment of the study of the variation of the functions and be the starting point for the treatment of optimization problems through the use of the derivative that is proposed at the levels of education mentioned above .

## **2. Theoretical foundation**

While it is true that mathematical knowledge structured and systematized under the scientific Interest of the discipline is the source to which it comes for incorporation in teaching, it suffers transformations that through transposition exercises didactics is arranged as an object of reachable teaching for students, whose purpose and objective is not that of discipline, but knowledge and its use in the resolution of problems of life, context and those of an intramathematical nature that encourage the development of creative, logical and heuristic skills, thus enhancing the rational use of school mathematical knowledge.

It should be appreciated that this does not mean a rupture with formalization, but that the emphasis on teaching of school mathematics is put in the intuitive processes during the construction of knowledge by students, starting from putting into play the application of various heuristic procedures, making arguments grow until they have access to formalization (Marmolejo & Moreno, 2018).

### **Heuristic resources**

Torres (2013) states that the heuristic instruction of mathematics, implies knowledge and conscious employment of three fundamental types of heuristic resources: the auxiliary heuristic means, the heuristic procedures and the general heuristic program. Here we emphasize the heuristic procedures, which constitute mental resources of search that allow orientation and to obtain the route of solution during the process of resolution of a mathematical problem. In this respect Jungk (1981) classifies these such heuristic procedures in principles, rules and heuristic strategies. The heuristic principles constitute suggestions for directly finding the idea of the main solution of resolution; they make it possible to determine, therefore, the means and the way of solution. Within these heuristic principles are identified: analogy, reduction and induction.

### **Software as a heuristic resource in the teaching and learning activity of mathematics**

Problems of teaching and learning mathematics at different levels, have motivated researchers as well as teachers towards the search for new tools to influence favorably on the teaching and learning processes of the discipline. Current trends in the teaching of mathematics have emphasized the importance of the use of educational technology as a tool that favor such processes. Researchers such as (Marmolejo & Campos, 2012; Arango, Gavirria & Valencia, 2015) highlight the importance of incorporating the use of technologies at work in the classroom, as these resources allow students to explore other environments that favor the resolution of teaching activities that are proposed to them and that, with only the use of pencil and paper, or the information of the Textbooks, class notes, among others, would be difficult for them to obtain. (Morales, Locia & Marmolejo, 2014; Morales, Locia & Salmerón, 2016) point out that Geogebra software is a heuristic resource that allows, via visualization and manipulation: to identify patterns of behaviors, properties and relationships between mathematical objects. They also emphasize that the heuristic procedures of analogy, reduction, induction and generalization are favored. These conclusions were identified from the analysis of proposals based on the software that researchers have done for the treatment of problems related to perimeters, areas, about the concept of median and the study of transformations of translation and rotation in the teaching of geometry in the pre-university student.

## **3. Methodology**

The strategy presented here took as a basis the methodologies on resolution of problems that are based on the historical-cultural approach and on the stages that structure it, describes the role of Geogebra software as a fundamental heuristic resource. In the development induction is favored by the dynamic-visual, then the generalization is allowed, a first scenario propiates the study of the behavior of curves through their tangents (mathematical models built), that they realize their stability and instability.

## Didactic strategy

The strategy described consists of the following steps: Identification of the problem, selection and formalization of the route, execution and control.

Stage 1. Identification of the problem. Some issues are proposed that favor the understanding of the problem, the identification of the information given and the requirement. To that end, some didactic impulses are fundamental, such as: what is the incognita, what are the data? What is the condition? Is the condition sufficient to determine the incognita? Is it insufficient? Redundant? Contradictory?

Stage 2. Selection and formalization of the via. Some way of proceeding is conceived, and the fundamental actions are described, that guide the activity towards the solution. For this stage, the following didactic impulses are fundamental: analogous problems, variants in the conditions, identification of concepts and properties in the situation, identification of the incognita, and it is a matter of remembering a problem that is familiar and has the same incognita or a similar incognita.

Stage 3. Execution and control. The indicated actions are carried out and the use of the variables, in addition, to verify the correct use of the variables, the structure of built model, the mathematical model, the relationship with the solution of the situation raised.

## Implementation of the strategy: analysis and discussions

**Problem 1.** *Divide into two parts, not necessarily equal, a segment of 7 units of length, in such a way that the product of the lengths of the two sub-segments that divide it is the maximum.*

Stage 1.

Depending on the conditions and the demand in the problem, it is asked to determine the length dimensions of the 7 main segment division length units; in such a way that the product of these lengths is the maximum. In a pen-and-paper environment, we would have to do infinite trials to construct the division sub-subsegments for each position of a point in the segment, then estimate the measurements of the lengths of these sub-segments and finally make the products. With the information determined by trial an estimation is achieved of the value of the maximum product, and therefore the appropriate lengths. After the identification of the requirement in the problem from trial procedures in a pen-and-paper environment, a pathway is now conceived, in a dynamic geometry environment that enables Geogebra software.

Stage 2.

- a. A 7 unit length segment is plotted. A point is located on it (not fixed) that runs through the segment and the two sub-segments of division are plotted.
- b. It is identified, that as the movable point runs through the main segment, the product of the lengths of the division sub-segments varies, and it is intuited that the proper position of the movable point must be in the vicinity of the center of the main segment, to make the product the maximum. This visualization is also supported by building and observing the behavior of the area of a rectangle whose dimensions are the lengths of the sub-segments. It is observed that the point approaches the center of the main segment, the figure tends to be a square, and the product associated with the area is the maximum.
- c. The area's geometric place is built as the movable point runs through the segment. At first the manual work is done with the software, and the general behavior is induced from this activity.
- d. Finally, it is discovered that the measurements of the lengths of the sub-segments that make their product the maximum are the same.

- e. Generalization principle: the mathematical model is constructed (quadratic function) of the situation. This is possible because the three points are known: (0, 0), (7, 0) and (a, c1) where a is a slider that runs through the  $\overline{AB}$  segment of 7 units, and c1 indicates the product of the lengths of the sub-segments.
- A point is located in the curve, and a tangent is constructed by that point. This activity is done in order to relate the intuitive work that is being performing with the tools offered by analytical geometry.
  - It is observed that as the abscissa of the point of tangency is approached by the left to the midpoint of the main segment, the tangent has an abscissa increased behavior, in a midpoint neighborhood the tangent tends to have a horizontal behavior, if the abscissa of the tangency point is on the right of the abscissa of the point in the main segment the tangent has a decreasing behavior. This identification is done by using the software, and a criterion is induced for the investigation of the maximum of a function, based on the criterion of the first derivative.it should be noted that the relevant process is the potential of the Geogebra software as the heuristic source that helps to rediscover mathematical knowledge and help to generalize behaviors of mathematical model associated with problem situations, such as that which is object of study of the work.

**Actions using software GeoGebra**

- a. Point C has been located on the segment, the division sub-segments are  $\overline{AC}$  and  $\overline{CB}$ . The product of the lengths of the sub-segments is indicated by  $c1 = (\overline{AC})(\overline{CB}) = 11.54$  as shown in figure 1.

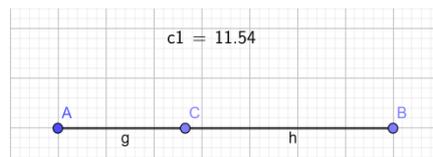


Figure 1. Source: own elaboration

- b. The value of c1 changes as point C traverses the main segment, as it was said in the previous stage, as the movable approaches the center, the product increase at its maximum value and tends to decrease as the point approaches the right end of the segment. With the idea of identifying and conjecturing on the segments of the sub-segments that satisfy the conditions of the problem, a rectangle was built when taking action pairs (consider that the pairs of measurements are variable because they are conditioned in this case by the route of point A in the segment  $\overline{BC}$ ), and thus by varying point A; the area measurements change as shown in Figure 2.

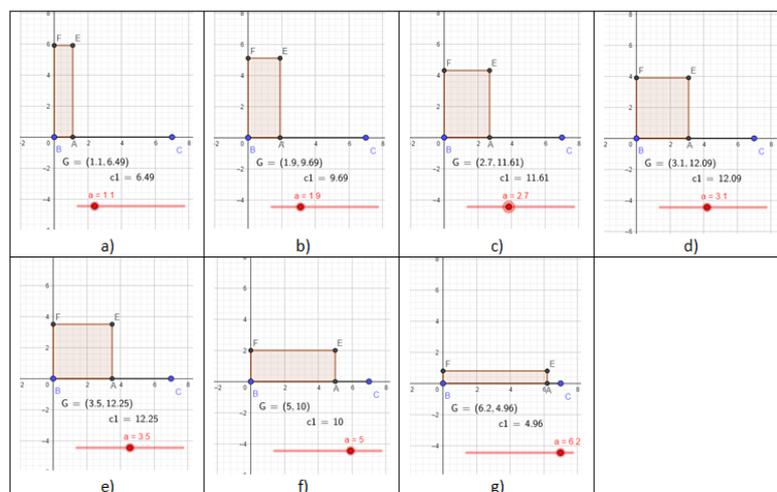
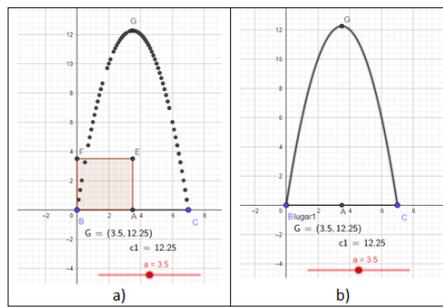


Figure 2. Source: own elaboration

c. Locus of the situation



In figure a) it is observed that by making the point A traverse the  $\overline{BC}$  segment it is described in the plane the locus. In figure b) a locus has been formally constructed, and it is identified that the greatest image is reached when point A is in a neighborhood of the center of the segment, and makes the figure tend to be a square.

d. with the previous analysis, it is concluded that the lengths of the sub-segments that make the product the maximum must be the same.

e. construction of the mathematical model (two-degree model) passing through three points:  $f(x) = -x^2 + 7x$ , its graph, and tangent to it, at the shown point, Figure 4.

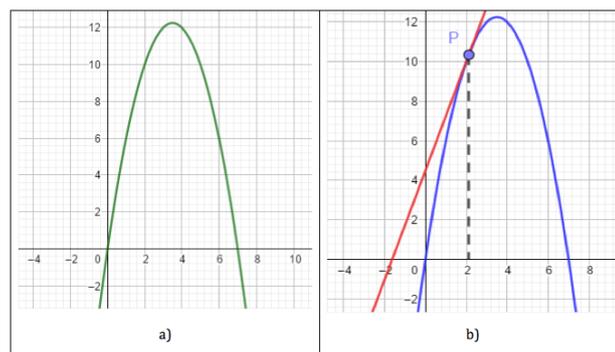


Figure 4. Source: own elaboration

- Tangent variation and identification of the relative maximum.

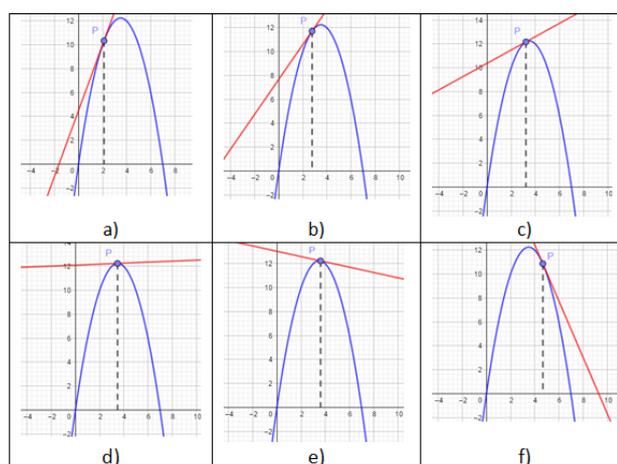


Figure 5. Source: own elaboration

In figure d), it is observed that the tangent tends to have a horizontal behavior when the abscissa of the tangency point tends to the vicinity of the center of the main segment. That is to say, that the behavior of the tangent

helps to discover and to systematize the maximum value of the function, therefore, the measurements of the lengths of the sub-segments that make the product maximum.

Stage 3.

To use the Geogebra software, the identification of the movable point was favored in the segment that allowed to determine the lengths of the two sub-segments of division, and ensure that the product is the maximum. We could also identify the behavior of the tangent over the curve that represented the area of the rectangle, as the point varied, it was observed that the tangency point changed position in the curve, and therefore the tangent. Of this last behavior, it was identified that in the maximum point in the abscissas grew and then decreased. This identification allowed two situations:

- i. To identify a first approach to the derivative criterion to investigate the relative maximum and/or minimum of a function.
- ii. To give input the analysis of the variation of the functions.

**Problem 2.** Given a circle of radius 5 units, a rectangle is inscribed on it. It is desired to determine the dimensions of rectangle of maximum area.

Stage 1.

In a pen-and-paper environment, they would be motivated to build particular cases and trial and error calculations can be made to intuit or pose a conjecture about the adequate dimensions of the rectangle that make the product the maximum. The essential thing at this stage is the identification of the requirement: to maximize the area, and it is identified that the information given is necessary.

Stage 2.

After proceeding by trial and error in a pen-and-paper environment (Stage 1) it is allowed to intuit that another way of conjecturing and approaching the solution of the problem is using the dynamic-visual resource offered by Geogebra software. Therefore, the following actions are carried out.

- a) By using the software, a 5 unit radius circle is built. A non-fixed rectangle is in inscribed on it.
- b) Manually with the software the dimensions of the rectangle are modified, being inscribed in the circle, and the changes are displayed to the extent of their area. The following figure shows some variations.

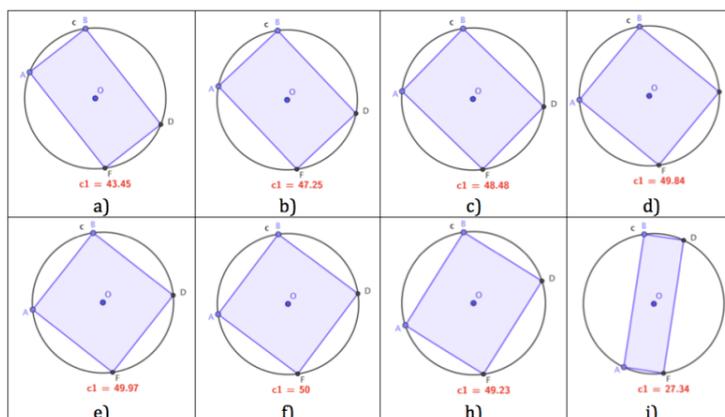


Figure 6. Source: own elaboration

c) It is observed that a good approximation to the solution of the problem is illustrated by figures 6), f1), and apparently it is a square (particular case of the rectangle). That is to say, using the software it is identified that the square in the figure of higher area, and the length if its side is 3.4 units. As shown in the following figure.

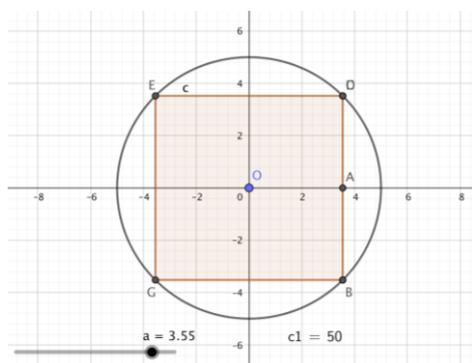


Figure 7. Source: own elaboration

Stage 3.

d) In order to systematize and test the conjecture

- The point of coordinates was graphed, the slider value vs. the value of the area of each rectangle as the slider is activated.
- The locus is constructed as the slider increases, making the journey from -5 to 5 units.
- The construction of the mathematical model is now sought to represent the situation, to this end, we use knowledge of geometry and trigonometry, essentially. Therefore, the mathematical model associated with

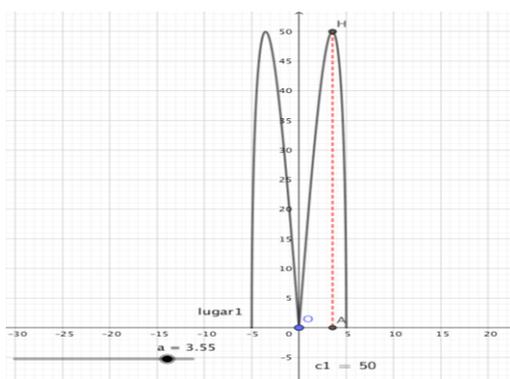


Figure 8. Source: own elaboration

- this situation is:  $A(x) = \pm 2x\sqrt{100 - 4x^2}$ .

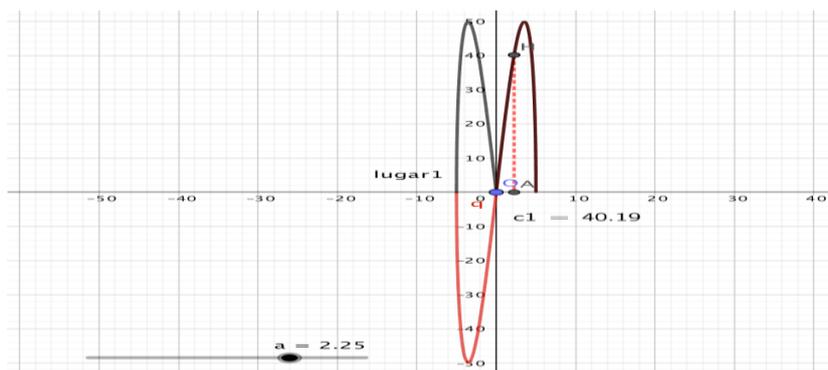


Figure 9. Source: own elaboration

With all intent we graphed on the curved, which describes the locus of the situation, and we can establish relationships, and explain some others. Taking into account that there are no areas of negative value, that makes the chart of the locus is above the horizontal axis. Actually, the branch of the graph of the built function, which is down of the horizontal axis is the reflection of the graph of the locus that is in the third quadrant. Well, the built graph can be traced to the tangent and its variation and analysis allows the identification of the role of the derivative in the investigation of the maximum, and therefore the determination of the appropriate lengths of the rectangle inscribed in the circumference, and that this has a maximum area.

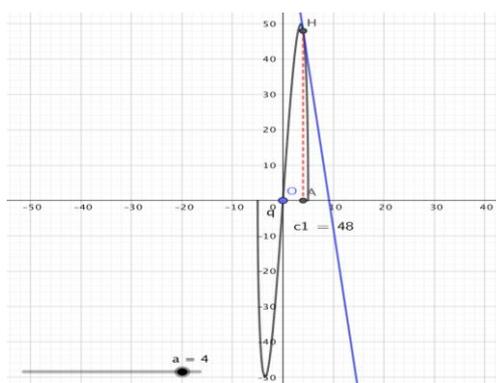


Figure 10. Source: own elaboration

When moving the point in the curve tangent variations are identified, this way, the maximum value of the graph (and/or minimum) the tangent is horizontal. In an analogous way to the study carried out in problem 1, the following situations are identified:

- i. An approximation to the criterion of the first derivative to investigate maximum and relative minimums.
- ii. The determination of the measure of the rectangle being inscribed in the circumference, has a maximum area, as it is the square.

### Conclusions

The proposal of theoretical-didactic cut here described for the introduction to problems that require the determination of the relative maximum through the GeoGebra software is essential to support the teaching activity of the teacher of pre-university student. The classical scheme close to the teacher for planning teaching and learning of variation of functions is the introduction of the concept of derivative, derivation formulas, and introduction of concepts such as: function increasing, decreasing, maximum, minimum and first and second derivative criteria for the investigation of relative extremes, among others. On rare occasions it raises the need

to ensure minimum starting conditions for the understanding and awareness of the importance of studying the problem of optimization.

In the opposite sense, the description of the strategy is fundamental for the understanding and meaning of mathematical concepts that allows the study of variation of functions, within them, the maximization, and in which, it is highlighted the usefulness of GeoGebra software as a heuristic resource, which through the activity of the dynamic-visual allows to rediscover knowledge, induction and generalization of behaviors.

## References

1. rango, J., Gavirria, D. y Valencia, A. (2015). Differential Calculus Teaching through Virtual Learning Objects in the Field of Management Sciences, *Procedia - Social and Behavioral Sciences*, 176, 412-418. doi.org/10.1016/j.sbspro.2015.01.490
2. Cuevas, C., y Delgado, M. (2016). ¿Por qué el concepto de función genera dificultad en el estudiante? *ReCalc*, 7, 108-119.
3. Delgado, M. (2013). Un problema con la concepción de la continuidad de una función. *El Cálculo y su Enseñanza*, 4, 27-44.
4. Jungk, W. (1981). *Conferencias sobre metodología de la enseñanza de la matemática 2. Segunda parte*. La Habana: Pueblo y Educación.
5. Marmolejo, J. E. y Campos, V. (2012). Pensamiento lógico matemático con scrath en el nivel básico. *Vínculos*, 9(1), 87-95.
6. Marmolejo, E. y Moreno G. (2018). De la intuición a la formalización. El caso de las Cónicas. Acercamiento a las Ciencias. UAGro. 153-179
7. Morales, A., Marmolejo, J. E., y Locia, E. (2014). El software GeoGebra: Un recurso heurístico en la resolución de problemas geométricos. *Premisa*, 16(63), 20-28.
8. Morales, A., Locia, E., y Salmerón, P. (2016). Recursos heurísticos para la actividad de enseñanza de las transformaciones geométricas en el nivel preuniversitario. *Atenas*, 3(35), 64-79.
9. Pineda, C. (2013). Una propuesta didáctica para la enseñanza del concepto de la derivada en el último grado de educación secundaria. (Tesis de Maestría). Universidad Nacional de Colombia. Bogotá, Colombia.
10. Ruiz, E. F., Gutiérrez, J. J., y Garay, L. I. (2018). Visualizando problemas de la derivada con aplicaciones en dispositivos móviles. *Innovación educativa*, 18(76), 39-67.
11. Dolores, C., García, M., Hernández, J. A., Sosa, L. (Eds.).(2013). La instrucción heurística en la formación de profesores de Matemáticas. México: Díaz de Santos.
12. Valencia, A. B. y Valenzuela, J. R. (2017). ¿A qué tipo de problemas matemáticos están expuestos los estudiantes de Cálculo? Un análisis de libros de texto. *Educación Matemática*, 29(3), 51-78. doi.org/10.24844/em2903.02