Three dimensional surfaces foliated by the equiform motion of pseudohyperbolic surfaces in $E^7$

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Abstract

In this paper we study three dimensional surfaces in $E^7$ generated by equiform motions of a pseudohyperbolic surface. The properties of these surfaces up to the first order are investigated. We prove that three dimensional surfaces in $E^7$ in general, is contained in a canal hypersurface, which is gained as envelope of a one-parametric set of 6-dimensional pseudohyperbolic. Finally we give an example.

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1. Introduction

An equiform transformation in the \( n \)-dimensional Euclidean space \( \mathbb{R}^n \) is an affine transformation whose linear part is composed from an orthogonal transformation and a homothetical transformation. Such an equiform transformation maps points \( x \in \mathbb{R}^n \) according to

\[
x \mapsto sAx + d, \quad s \in \mathbb{R}^+, \quad d \in \mathbb{R}^n.
\]

The number \( s \) is called the scaling factor. An equiform motion is defined if the parameters of (1), including \( s \), are given as functions of a time parameter \( t \). Then a smooth one-parameter equiform motion moves a point \( x \) via \( x(t) = s(t)A(t)x(t) + d(t) \). The kinematic corresponding to this transformation group is called equiform kinematic. See [2]. Recently, the equiform kinematic geometry has been used in computer vision and reverse engineering of geometric models such as the problem of reconstruction of a computer model from an existing object which is known (a large number of) data points on the surface of the technical object [9, 11]. In [8], they studied two-parameter spatial motions \( M(\lambda, \mu) \) in three dimensional Euclidean space from a differential geometric point of view, which (up to the second order) instantaneously move on locally one-dimensional point paths. In [1, 12], they studied first order properties of cyclic surfaces generated by the equiform motions in five dimensional Euclidean space and semi-Euclidean space.

In Minkowski (semi-Euclidean) space \( \mathbb{E}^3 \) with scalar product \( <x, y> = -x_1y_1 + x_2y_2 + x_3y_3 \) the pseudosphere or Lorentz sphere and the pseudohyperbolic surface play the same role as sphere in Euclidean space. Lorentz sphere of radius \( r > 0 \) in \( \mathbb{E}^3 \) is the quadric

\[
S^2(r) = \{ p \in \mathbb{E}^3 : < p, p > = r^2 \}.
\]

This surface is timelike and is the hyperboloid of one sheet \( -x_1^2 + x_2^2 + x_3^2 = r^2 \) which is obtained by rotating the hyperbola \( -x_1^2 + x_2^2 = r^2 \) in the plane \( x_2 = 0 \) with respect to the \( x_1 \)-axis. The pseudohyperbolic surface is the quadratic

\[
H_0^2(r) = \{ p \in \mathbb{E}^3 : < p, p > = -r^2 \}.
\]

This surface is spacelike and is the hyperboloid of two sheet \( -x_1^2 + x_2^2 + x_3^2 = -r^2 \) which is obtained by rotating the hyperbola \( x_1^2 - x_2^2 = r^2 \) in the plane \( x_2 = 0 \) with respect to the \( x_1 \)-axis [10].

In this paper we consider the equiform motions of a pseudohyperbolic surface \( k_s \) in \( \mathbb{E}^n \). The point paths of the pseudohyperbolic surface, generate three-dimensional surface, contains the positions of the starting pseudohyperbolic surface \( k_0 \). The first order properties of these surfaces for the points of these pseudohyperbolic surfaces have been studied for arbitrary dimensions \( n \geq 3 \). We restrict our considerations to dimension \( n = 7 \) because, at any moment the infinitesimal transformations of the motion maps the points of the pseudohyperbolic surface \( k_s \) to the velocity vectors, whose end points will form an affine image of \( k_s \) (in general a pseudohyperbolic surface \( k_\phi \)). Both these surfaces are space and therefore span a subspace \( W \) of \( \mathbb{E}^n \) with \( n \leq 7 \). Moreover, we show that any three-dimensional surfaces in \( \mathbb{E}^7 \) is in general contained in a canal hypersurface, which is gained as envelope of a one-parametric set of 6-dimensional pseudosphere.

2. Local study in canonical frames

Consider a unit pseudohyperbolic surface \( k_s \) in the space \( \pi_s = [x_1, x_2, x_3] \) centered at the origin represented by

\[
x(\theta, \phi) = (\cosh \theta, \sinh \theta \sin \phi, \sinh \theta \cos \phi, 0, 0, 0) \quad \text{in} \quad \mathbb{R} \text{ and } \phi \in [0, 2\pi],
\]

the general representation of the motion of three-dimensional surface in \( \mathbb{E}^7 \) foliated by two-dimensional pseudohyperbolic surface is given by
\[ X(t, \theta, \phi) = s(t)A(t)x(\theta, \phi) + d(t), t \in \mathbb{R} \]

where \( d(t) = (b_1(t), b_2(t), b_3(t), b_4(t), b_5(t), b_6(t), b_7(t))^T \) describes the position of the origin of \( \Sigma^o \) at the time \( t \), \( A(t) = (a_{ij}(t)) \), \( 1 \leq i, j \leq 7 \) is a semi orthogonal matrix and \( s(t) \) provides the scaling factor of the moving system. Moreover we assume that all involved functions are of class \( C^1 \). Using Taylor’s expansion, up to the first order then the representation of the motion is given by

\[ X(t, \theta, \phi) = \{s(0)A(0) + [s(0)A(0) + s(0)\dot{A}(0)]t\}x(\theta, \phi) + d(0) + t\dot{d}(0), \]

where \( \dot{} \) denotes differentiation with respect to time \( (t = 0) \). As an equiform motion has an invariant point, we can assume without loss of generality that the moving frame \( E^7 \) and fixed frame \( \Sigma \) coinciding at the zero position \( (t = 0) \), then we have

\[ A(0) = I, \quad s(0) = 1 \quad \text{and} \quad d(0) = 0, \]

thus

\[ X(t, \theta, \phi) = [I + (sI + \Omega)t]x(\theta, \phi) + t\dot{d}', \]

where \( \Omega = \dot{A}(0) = (\omega_k), k = 1, 2, 3, ..., 21 \) is a semi skew symmetric matrix. In this paper all values of \( s, b_i \) and their derivatives are computed at \( t = 0 \) and for simplicity, we write \( s' \) and \( b'_i \) instead of \( \dot{s}(0) \) and \( \dot{b}_i(0) \) respectively. In these frames, the representation of \( X(t, \theta, \phi) \) is given by

\[
\begin{pmatrix}
X_1 \\
X_2 \\
X_3 \\
X_4 \\
X_5 \\
X_6 \\
X_7
\end{pmatrix}
= \begin{pmatrix}
1 + s't & \omega_1 & \omega_2 & \omega_3 & \omega_4 & \omega_5 & \omega_6 \\
\omega_1 & 1 + s't & \omega_7 & \omega_8 & \omega_9 & \omega_{10} & \omega_{11} \\
\omega_2 & -\omega_7 & 1 + s't & \omega_{12} & \omega_{13} & \omega_{14} & \omega_{15} \\
\omega_3 & -\omega_8 & -\omega_{12} & 1 + s't & \omega_{16} & \omega_{17} & \omega_{18} \\
\omega_4 & -\omega_9 & -\omega_{13} & -\omega_{16} & 1 + s't & \omega_{19} & \omega_{20} \\
\omega_5 & -\omega_{10} & -\omega_{14} & -\omega_{17} & -\omega_{19} & 1 + s't & \omega_{21} \\
\omega_6 & -\omega_{11} & -\omega_{15} & -\omega_{18} & -\omega_{20} & -\omega_{21} & 1 + s't
\end{pmatrix}
\begin{pmatrix}
cosh \phi \\
\sinh \phi \sin \phi \\
\sinh \phi \cos \phi \\
\sin \phi \\
\cosh \phi \\
\sin \phi \\
\cosh \phi
\end{pmatrix}
\begin{pmatrix}
b'_1 \\
b'_2 \\
b'_3 \\
b'_4 \\
b'_5 \\
b'_6 \\
0
\end{pmatrix}
\]

or in the equivalent form

\[
\begin{pmatrix}
X_1 \\
X_2 \\
X_3 \\
X_4 \\
X_5 \\
X_6 \\
X_7
\end{pmatrix}
= t
\begin{pmatrix}
1 + s't \\
\omega_1 \\
\omega_2 \\
\omega_3 \\
\omega_4 \\
\omega_5 \\
\omega_6
\end{pmatrix}
+ \begin{pmatrix}
cosh \theta \\
\sinh \theta \sin \phi \\
\sinh \theta \cos \phi \\
\sin \theta \sin \phi \\
\sin \theta \cos \phi \\
\sin \theta \sin \phi \\
\sin \theta \cos \phi
\end{pmatrix}
\begin{pmatrix}
b'_1 \\
b'_2 \\
b'_3 \\
b'_4 \\
b'_5 \\
b'_6 \\
0
\end{pmatrix}
\]

\[ = \tilde{t}b + \tilde{a}_x \cosh \theta + \tilde{a}_y \sinh \theta \sin \phi + \tilde{a}_z \sinh \theta \cos \phi. \]
For any fixed \( t \) in the above expression (3), we generally gain an elliptical hyperboloid for \( \theta \in \mathbb{R} \) and \( \phi \in [0,2\pi] \) centered at the point \( t(b'_1,b'_2,b'_3,b'_4,b'_6,b'_7) \). The latter elliptical hyperboloid turns to a two-dimensional pseudohyperbolic surface if \( \bar{a}_0, \bar{a}_1 \) and \( \bar{a}_2 \) form an orthogonal basis. This gives the conditions

\[
\begin{align*}
\omega_2^2 + \omega_3^2 + \omega_4^2 + \omega_5^2 + \omega_6^2 + \omega_7^2 &= -\omega_2^2 + \omega_3^2 + \omega_4^2 + \omega_5^2 + \omega_6^2 + \omega_7^2 \\
&= -\omega_2^2 + \omega_3^2 + \omega_4^2 + \omega_5^2 + \omega_6^2 + \omega_7^2 \\
&= 0
\end{align*}
\]

and

\[
\begin{align*}
\omega_2^2 + \omega_3^2 + \omega_4^2 + \omega_5^2 + \omega_6^2 + \omega_7^2 &= \omega_2^2 - \omega_3^2 - \omega_4^2 - \omega_5^2 - \omega_6^2 - \omega_7^2 \\
&= \omega_2^2 - \omega_3^2 - \omega_4^2 - \omega_5^2 - \omega_6^2 - \omega_7^2 \\
&= a,
\end{align*}
\]

where \( a \in \mathbb{R}^+ \). Thus we get the following equation of the pseudohyperbolic space

\[
\sum_{i=1}^{7} \varepsilon_i (x_i - t b'_i)^2 = at^2 - (1 + s't)^2,
\]

where \( \varepsilon_i = -1, \varepsilon_j = 1, j = 2,3,4,5,6,7 \). The orthogonal projection of these elliptical hyperboloid \( (t = \text{const}.) \) in (3)) on the space of the starting pseudohyperbolic surface \( \pi_s = [x_1,x_2,x_3] \), is

\[
\begin{pmatrix}
X_1 \\
X_2 \\
X_3
\end{pmatrix} = t \begin{pmatrix}
b'_1 \\
b'_2 \\
b'_3
\end{pmatrix} + \begin{pmatrix}
1 + s't \\
t \omega_1 \\
t \omega_2
\end{pmatrix} \cosh \theta + \begin{pmatrix}
t \omega_3 \\
1 + s't \\
t \omega_4
\end{pmatrix} \sinh \theta \sin \phi + \begin{pmatrix}
t \omega_5 \\
t \omega_6 \\
1 + s't
\end{pmatrix} \sinh \theta \cos \phi.
\]

This equation generalizes in five dimension that happens for \( \phi = 0 \). Namely, if \( \phi = 0 \) the orthogonal projection of the elliptical hyperboloid in equation (4) on the space \( [x_1,x_3] \) is

\[
\begin{pmatrix}
X_1 \\
X_3
\end{pmatrix} = t \begin{pmatrix}
b'_1 \\
b'_3
\end{pmatrix} + \begin{pmatrix}
1 + s't \\
t \omega_2
\end{pmatrix} \cosh \theta + \begin{pmatrix}
t \omega_2 \\
1 + s't
\end{pmatrix} \sinh \theta.
\]

This gives Lorentzian circles centered at \( (tb'_1, tb'_3) \) and radii by \( \sqrt{t^2 \omega_2^2 - (1 + s't)^2} \).

**Corollary 2.1**

The projection of the ruled surface of tangent to \( k_1 \) into the original space will give a three-dimensional surface in \( \mathbb{E}^3 \), which is foliated by elliptical hyperboloids. Now from (4) we have

\[
X(t,\theta,\phi) = \begin{pmatrix}
1 + s't \\
t \omega_1 \\
t \omega_2 \\
t \omega_3 \\
t \omega_4 \\
t \omega_5 \\
t \omega_6 \\
t \omega_7
\end{pmatrix} \begin{pmatrix}
\cosh \theta \\
\sinh \theta \sin \phi \\
\sinh \theta \cos \phi
\end{pmatrix} + t \begin{pmatrix}
b'_1 \\
b'_2 \\
b'_3
\end{pmatrix},
\]
and the first partial derivatives are

\[
X_i = \begin{pmatrix}
 b'_1 \\
 b'_2 \\
 b'_3 \\
\end{pmatrix} + \begin{pmatrix}
 s' & \omega_i & \omega_2 \\
 \omega_1 & s' & \omega_7 \\
 -\omega_2 & s' & \omega_7 \\
\end{pmatrix} \begin{pmatrix}
 \cosh \theta \\
 \sinh \theta \sin \phi \\
 \sinh \theta \cos \phi \\
\end{pmatrix},
\]

\[
X_\theta = (\sinh \theta, \cosh \theta \sin \phi, \cosh \theta \cos \phi)^T,
\]

\[
X_\phi = (0, \sinh \theta \cos \phi, -\sinh \theta \sin \phi)^T.
\]

Then the linearly dependent points

\[
\sinh \theta[-s' - b'_1 \cosh \theta + b'_2 \sinh \theta \sin \phi + b'_3 \sinh \theta \cos \phi] = 0,
\]

we get

\[
\sinh \theta[-s' + <d', x(\theta, \phi)>] = 0.
\]

The latter equation characterizes the instantaneous curve of contact.

**3. Tangent pseudosphere of three-dimensional surface in \( \mathbb{E}^7 \)**

In this section we will show that at any instant \( t \) there exists a pseudosphere \( K(t) \), which is tangent to a given three-dimensional surface \( (2) \) in all points of the instantaneous position \( k(t) \) of the pseudohyperbolic surface \( k_\theta \).

Without loss of generality we investigate the situation at the zero position. Any pseudosphere \( K_\theta \) which is tangent to the given three-dimensional surface \( (2) \) along \( k_\theta \) has to contain \( k_\theta \), hence the center of \( K_\theta \) has coordinates \( (0,0,0,m_4,m_5,m_6,m_7) \) with \( m_4,m_5,m_6,m_7 \in \mathbb{R} \). On the other hand since \( K_\theta \) has to be tangent to all velocity vectors of the motion, the center of \( K_\theta \) has to lie in each of the hyperplanes through the points of \( k(t) \) orthogonal to these velocity vectors. This gives us the additional condition

\[
m_4(b'_4 + \omega_1 \cosh \theta - \omega_8 \sinh \theta \sin \phi - \omega_{12} \sinh \theta \cos \phi) \\
+ m_5(b'_5 + \omega_4 \cosh \theta - \omega_9 \sinh \theta \sin \phi - \omega_{13} \sinh \theta \cos \phi) \\
+ m_6(b'_6 + \omega_6 \cosh \theta - \omega_{10} \sinh \theta \sin \phi - \omega_{14} \sinh \theta \cos \phi) \\
+ m_7(b'_7 + \omega_7 \cosh \theta - \omega_{11} \sinh \theta \sin \phi - \omega_{15} \sinh \theta \cos \phi) = -s' - b'_1 \cosh \theta + b'_2 \sinh \theta \sin \phi + b'_3 \sinh \theta \cos \phi.
\]

By comparing the coefficients of \( \{1, \cosh \theta, \sinh \theta \sin \phi, \sinh \theta \cos \phi\} \) in (5), we have the system of linear equations

\[
BM = H,
\]

where

\[
B = \begin{pmatrix}
 b'_4 & b'_5 & b'_6 & b'_7 \\
 \omega_4 & \omega_5 & \omega_6 & \omega_7 \\
 \omega_8 & \omega_9 & \omega_{10} & \omega_{11} \\
 \omega_{12} & \omega_{13} & \omega_{14} & \omega_{15} \\
\end{pmatrix},
\]

\[
M = \begin{pmatrix}
 m_4 \\
 m_5 \\
 m_6 \\
 m_7 \\
\end{pmatrix}
\]

and

\[
H = \begin{pmatrix}
 -s' \\
 -b'_1 \\
 -b'_2 \\
 -b'_3 \\
\end{pmatrix}.
\]

If \( B \) is a regular matrix, we get
\[ M = B^{-1}H. \] (7)

Therefore, we have the following theorem:

**Theorem 3.1**

**Definition 3.1** Canal hypersurfaces in \( \mathbb{E}^n \) are envelope hypersurfaces of one-parametric sets of pseudospheres.

Therefore, we have the following theorem

**Theorem 3.2**

**3.1 The singular cases**

If the system of equations (6) is singular, we have many cases:

**Case 1.** \( rank(B) = rank(B \setminus H) = 3 \). In this case, we have a one-parametric set of pseudospheres whose centers fulfil a straight line in the \( x_4, x_5, x_6, x_7 \) – space

\[ M = (0, 0, 0, m_4, x_5(m_4), x_6(m_4), x_7(m_4)), \]

where

\[
x_5(m_4) = \frac{1}{\Delta} \left[ (\omega_5 \omega_1 - \omega_6 \omega_0) (s' + b'_4 m_4) + (b'_6 \omega_1 - b'_5 \omega_0) (b'_4 + \omega_3 m_4) \right. \\
+ \left. (b'_5 \omega_3 - b'_6 \omega_5) (b'_4 + \omega_3 m_4) \right],
\]

\[
x_6(m_4) = \frac{1}{\Delta} \left[ (\omega_6 \omega_3 - \omega_2 \omega_4) (s' + b'_4 m_4) + (b'_5 \omega_3 - b'_6 \omega_4) (b'_4 + \omega_3 m_4) \right. \\
+ \left. (b'_6 \omega_4 - b'_5 \omega_6) (b'_4 + \omega_3 m_4) \right],
\]

\[
x_7(m_4) = \frac{1}{\Delta^*} \left[ (\omega_4 \omega_1 - \omega_5 \omega_2) (s' + b'_4 m_4) + (b'_5 \omega_1 - b'_6 \omega_2) (b'_4 + \omega_3 m_4) \right. \\
+ \left. (b'_6 \omega_2 - b'_5 \omega_4) (b'_4 + \omega_3 m_4) \right],
\]

where

\[
\Delta = b'_4 (\omega_5 \omega_1 - \omega_6 \omega_0) + b'_6 (\omega_6 \omega_3 - \omega_2 \omega_4) + b'_5 (\omega_4 \omega_1 - \omega_5 \omega_2),
\]

\[
\Delta^* = b'_4 (\omega_5 \omega_1 - \omega_6 \omega_0) + b'_6 (\omega_6 \omega_3 - \omega_2 \omega_4) + b'_5 (\omega_4 \omega_1 - \omega_5 \omega_2),
\]

with arbitrary \( m_4 \in \mathbb{R} \). Thus, we get a straight line of possible centers.

**Case 2.** \( rank(B) = rank(B \setminus H) = 2 \). In this case, we have a two-parametric set of pseudospheres whose centers fulfil a surface in \( x_4, x_5, x_6, x_7 \) – space

\[ M = (0, 0, 0, m_4, m_5, x_5(m_4, m_5), x_6(m_4, m_5)), \]

where

\[
x_6(m_4, m_5) = \frac{m_4 (b'_5 \omega_3 - b'_6 \omega_5) + m_5 (b'_6 \omega_3 - b'_5 \omega_6) + (s' \omega_5 - b'_5 b'_6)}{b'_5 \omega_3 - b'_6 \omega_5},
\]
\[ x_7(m_4, m_5) = \frac{m_4(b'_1\omega_b - b'_1\omega_3) + m_5(b'_1\omega_b - b'_1\omega_4) - (s'\omega_b - b'b'_1)}{b'_1\omega_b - b'_1\omega_6}, \]

with arbitrary \( m_4, m_5 \in \mathbb{R} \). Thus, we get a surface of possible centers.

**Case 3.** \( \text{rank}(B) = \text{rank}(B \setminus H) = 1 \). In this case, we have a hyperplane of possible centers. **Case 4.** \( \text{rank}(B) = 3 \neq \text{rank}(B \setminus H) \). In this case we assume

\[
\frac{\omega_8}{\omega_{12}} = \frac{\omega_9}{\omega_{13}} = \frac{\omega_{10}}{\omega_{14}} = \frac{\omega_{11}}{\omega_{15}} = \lambda, \quad \frac{b'_2}{b'_3} \neq \lambda.
\]

By using the homogenous coordinates

\[
m_b = \Delta = 0, \quad m_1 = 0, \quad m_2 = 0, \quad m_3 = 0,
\]

\[
m_4 = (b'_2 - b'_3)[b'_1(\omega_b\omega_{14} - \omega_3\omega_{15}) + b'_1(\omega_b\omega_6 - \omega_4\omega_{13}) + b'_1(\omega_3\omega_{13} - \omega_4\omega_{14})]
\]

\[
m_5 = (b'_2 - b'_3)[b'_4(\omega_3\omega_{15} - \omega_6\omega_{14}) + b'_4(\omega_3\omega_{12} - \omega_6\omega_{13}) + b'_4(\omega_3\omega_{14} - \omega_6\omega_{12})]
\]

\[
m_6 = (b'_2 - b'_3)[b'_5(\omega_6\omega_{13} - \omega_3\omega_{14}) + b'_5(\omega_6\omega_{12} - \omega_3\omega_{15}) + b'_5(\omega_6\omega_{14} - \omega_3\omega_{12})]
\]

\[
m_7 = (b'_2 - b'_3)[b'_6(\omega_1\omega_{14} - \omega_6\omega_{13}) + b'_6(\omega_1\omega_{12} - \omega_6\omega_{14}) + b'_6(\omega_1\omega_{13} - \omega_6\omega_{15})]
\]

Then the centers of the pseudospheres are an ideal point (point at infinity). The corresponding pseudospheres degenerates into a hyperplane.

**Case 5.** \( \text{rank}(B) = 2 \neq \text{rank}(B \setminus H) \). In this case we assume

\[
\frac{\omega_8}{\omega_{12}} = \frac{\omega_9}{\omega_{13}} = \frac{\omega_{10}}{\omega_{14}} = \frac{\omega_{11}}{\omega_{15}} = \lambda, \quad \frac{b'_2}{b'_3} \neq \lambda,
\]

\[
\frac{\omega_8}{\omega_{12}} = \frac{\omega_9}{\omega_{13}} = \frac{\omega_{10}}{\omega_{14}} = \frac{\omega_{11}}{\omega_{15}} = \mu, \quad \frac{b'_2}{b'_3} \neq \mu.
\]

Using the homogenous coordinates

\[
m_c = \Delta = 0, \quad m_1 = 0, \quad m_2 = 0, \quad m_3 = 0
\]

\[
m_4 = (\lambda b'_2 - \lambda \mu b'_3)[b'_5(\omega_1\omega_{14} - \omega_6\omega_{13}) + b'_5(\omega_1\omega_{12} - \omega_6\omega_{15}) + b'_5(\omega_1\omega_{13} - \omega_6\omega_{14})]
\]

\[
m_5 = (\lambda b'_2 - \lambda \mu b'_3)[b'_4(\omega_3\omega_{15} - \omega_6\omega_{14}) + b'_4(\omega_3\omega_{12} - \omega_6\omega_{13}) + b'_4(\omega_3\omega_{14} - \omega_6\omega_{12})]
\]
\[ m_6 = (\lambda b'_2 - \lambda \mu b'_3)(b'_4(\omega_1 \omega_3 - \omega_9 \omega_{15}) + b'_5(\omega_8 \omega_{15} - \omega_9 \omega_{12}) + b'_7(\omega_9 \omega_{12} - \omega_8 \omega_{13})) \]

\[ m_7 = (\lambda b'_2 - \lambda \mu b'_3)(b'_4(\omega_3 \omega_4 - \omega_{16} \omega_1) + b'_5(\omega_4 \omega_2 - \omega_6 \omega_{14}) + b'_7(\omega_8 \omega_{13} - \omega_9 \omega_{12})) \]

Then we have the same result as in case 4.

**Case 6.** \( \text{rank}(B) = 1 \neq \text{rank}(B \setminus H) \). In this case the centers of the possible pseudospheres tends to a straight line at infinity. The corresponding pseudospheres degenerate and formed a pencil of hyperplanes. They contain 4-dimensional subspaces, which contains the given starting pseudohyperbolic surface \( k \), and the corresponding velocity vectors. This leads directly to the well known result in \( E^3 \), that there is in general will be no series of pseudospheres tangent to the three-dimensional surfaces.

4. **Curve of centers of the pseudospheres**

Now, we consider \( t \) is varying and in this section, we will determine the centers of pseudospheres which contain a pseudohyperbolic surface \( k(t) \) and are tangent to all tangent planes \( \tau(t, \theta, \phi) \) of the three-dimensional surface (2). Let \( a_i(t), i = 1, 2, ..., 7 \) are the column vectors of the matrix \( A(t) \), then (2) can be represented in the following way

\[ X(t, \theta, \phi) = s(t)[a_1(t) \cosh \theta + a_2(t) \sinh \theta \sin \phi + a_5(t) \sinh \theta \cos \phi] + d(t), \quad (8) \]

where \( d(t) \) is the center of the moving pseudohyperbolic surface and \( a_1(t), a_2(t), a_5(t) \) are three orthogonal vectors in the space of the moving pseudohyperbolic surface. The velocity vectors of the points of the sphere are given by

\[ X'(t, \theta, \phi) = [s'(t)a_1(t) + s(t)a'_1(t)] \cosh \theta + [s'(t)a_2(t) + s(t)a'_2(t)] \sinh \theta \sin \phi + [s'(t)a_5(t) + s(t)a'_5(t)] \sinh \theta \cos \phi + d'(t). \quad (9) \]

The equation of the hyperplanes orthogonal to such a path is

\[ Y^T X'(t, \theta, \phi) = X^T (t, \theta, \phi) X'(t, \theta, \phi), \]

where \( Y = (y_1, y_2, y_3, y_4, y_5, y_6, y_7)^T \) is the position vector of an arbitrary point \( Y \) in the hyperplane. The scalar product in the above equation is Lorentz metric. According to the inner product this equation is

\[ Y^T \varepsilon X'(t, \theta, \phi) = X^T (t, \theta, \phi) \varepsilon X'(t, \theta, \phi), \quad (10) \]

where \( \varepsilon = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \]

is the sign matrix. Substitution of equations (8) and (9) into (10), yields
\[
Y^T \varepsilon [s'(t) a_i(t) + s(t) a'_i(t)] \cosh \theta + Y^T \varepsilon [s'(t) a_z(t) + s(t) a'_z(t)] \sinh \theta \sin \phi \\
+ Y^T \varepsilon [s'(t) a_t(t) + s(t) a'_t(t)] \sinh \theta \cos \phi + Y^T \varepsilon a' \phi(t) \\
= (s(t) a'_t(t)) \cosh \theta + s(t) a'_z(t) \sinh \theta \sin \phi + s(t) a'_t(t) \sinh \theta \cos \phi + d^T(t) \\
(11)
\]

\[
\varepsilon [s'(t) a_i(t) + s(t) a'_i(t)] \cosh \theta + [s'(t) a_z(t) + s(t) a'_z(t)] \sinh \theta \sin \phi \\
+ [s'(t) a_t(t) + s(t) a'_t(t)] \sinh \theta \cos \phi + d'(t)).
\]

Since \( A^T A = \varepsilon \) and \( A' A' \) is a skew symmetric matrix, let \( e_k(t) = a'_k(t) \phi(t), h_k(t) = a'_k(t) \phi(t) \) and \( \ell_k(t) = a_k(t) \phi(t), k = 1, 2, 3 \). Then by comparing the coefficients of

\[
\{1, \cosh \theta, \sinh \theta \sin \phi, \sinh \theta \cos \phi \}
\]

in (11), we obtain

\[
\sum_{i=1}^{7} \varepsilon_{,i} b_i'(t) = \sum_{i=1}^{7} \varepsilon_{,i} b_i'(t) - s(t)s'(t), \\
\sum_{i=1}^{7} \varepsilon_{,i} y_{,i} a_i(t) + s(t) \sum_{i=1}^{7} \varepsilon_{,i} y_{,i} a_i'(t) = s(t)(e_i(t) + h_i(t)) + s'(t)\ell_i(t), \\
\sum_{i=1}^{7} \varepsilon_{,i} y_{,i} a_{,i} a_i(t) + s(t) \sum_{i=1}^{7} \varepsilon_{,i} y_{,i} a_{,i} a_i'(t) = s(t)(e_i(t) + h_i(t)) + s'(t)\ell_i(t), \\
\sum_{i=1}^{7} \varepsilon_{,i} y_{,i} a_{,i} a_i(t) + s(t) \sum_{i=1}^{7} \varepsilon_{,i} y_{,i} a_{,i} a_i'(t) = s(t)(e_i(t) + h_i(t)) + s'(t)\ell_i(t).
\]

where \( \varepsilon_1 = -1, \varepsilon_j = 1, j = 2, 3, 4, 5, 6, 7 \). We know from the initial position, that the hyperplanes of the three-dimensional surfaces contain a point \( m(t) \) for any \( t \) and \( \forall \theta, \phi \) such that \( m(t) = (0, 0, 0, m_4(t), m_5(t), m_6(t), m_7(t)) \) is the center of this pseudosphere, then from (12), one can find

\[
FM = Q,
\]

where

\[
F = \begin{pmatrix} 
\begin{vmatrix} 
\varepsilon_{,i} \phi(t) \\
s'(t) a_{41} + s(t) a'_{41}(t) \\
s'(t) a_{42} + s(t) a'_{42}(t) \\
s'(t) a_{43} + s(t) a'_{43}(t)
\end{vmatrix} & 
\begin{vmatrix} 
\varepsilon_{,i} \phi(t) \\
s'(t) a_{51} + s(t) a'_{51}(t) \\
s'(t) a_{52} + s(t) a'_52(t) \\
s'(t) a_{53} + s(t) a'_{53}(t)
\end{vmatrix} & 
\begin{vmatrix} 
\varepsilon_{,i} \phi(t) \\
s'(t) a_{61} + s(t) a'_{61}(t) \\
s'(t) a_{62} + s(t) a'_{62}(t) \\
s'(t) a_{63} + s(t) a'_{63}(t)
\end{vmatrix} \\
\begin{vmatrix} 
\varepsilon_{,i} \phi(t) \\
s'(t) a_{71} + s(t) a'_{71}(t) \\
s'(t) a_{72} + s(t) a'_{72}(t) \\
s'(t) a_{73} + s(t) a'_{73}(t)
\end{vmatrix}
\end{pmatrix}
\]

\[
M = \begin{pmatrix} 
m_4(t) \\
m_5(t) \\
m_6(t) \\
m_7(t)
\end{pmatrix}
\quad \text{and} \quad 
Q = \begin{pmatrix} 
\sum_{i=1}^{7} \varepsilon_{,i} b_i'(t) - s(t)s'(t) \\
\sum_{i=1}^{7} \varepsilon_{,i} b_i'(t) - s(t)s'(t) \\
\sum_{i=1}^{7} \varepsilon_{,i} b_i'(t) - s(t)s'(t) \\
\sum_{i=1}^{7} \varepsilon_{,i} b_i'(t) - s(t)s'(t)
\end{pmatrix}
\begin{pmatrix} 
s(t)(e_1(t) + h_1(t)) + s'(t)\ell_1(t) \\
s(t)(e_i(t) + h_i(t)) + s'(t)\ell_i(t) \\
s(t)(e_2(t) + h_2(t)) + s'(t)\ell_2(t) \\
s(t)(e_i(t) + h_i(t)) + s'(t)\ell_i(t)
\end{pmatrix}.
\]

If \( F \) is a regular matrix, we get

\[
M = F^{-1}Q.
\]
Therefor, the coordinates of the centers of the pseudospheres in the fixed frame at any instant \( t \) are given by

\[
\begin{pmatrix}
M_1 \\
M_2 \\
M_3 \\
M_4 \\
M_5 \\
M_6 \\
M_7
\end{pmatrix}
= s(t) A(t)
\begin{pmatrix}
0 \\
0 \\
0 \\
m_4(t) \\
m_5(t) \\
m_6(t) \\
m_7(t)
\end{pmatrix}
+ d(t).
\] (15)

**Theorem 4.1**

**Example 1**

**References**


